

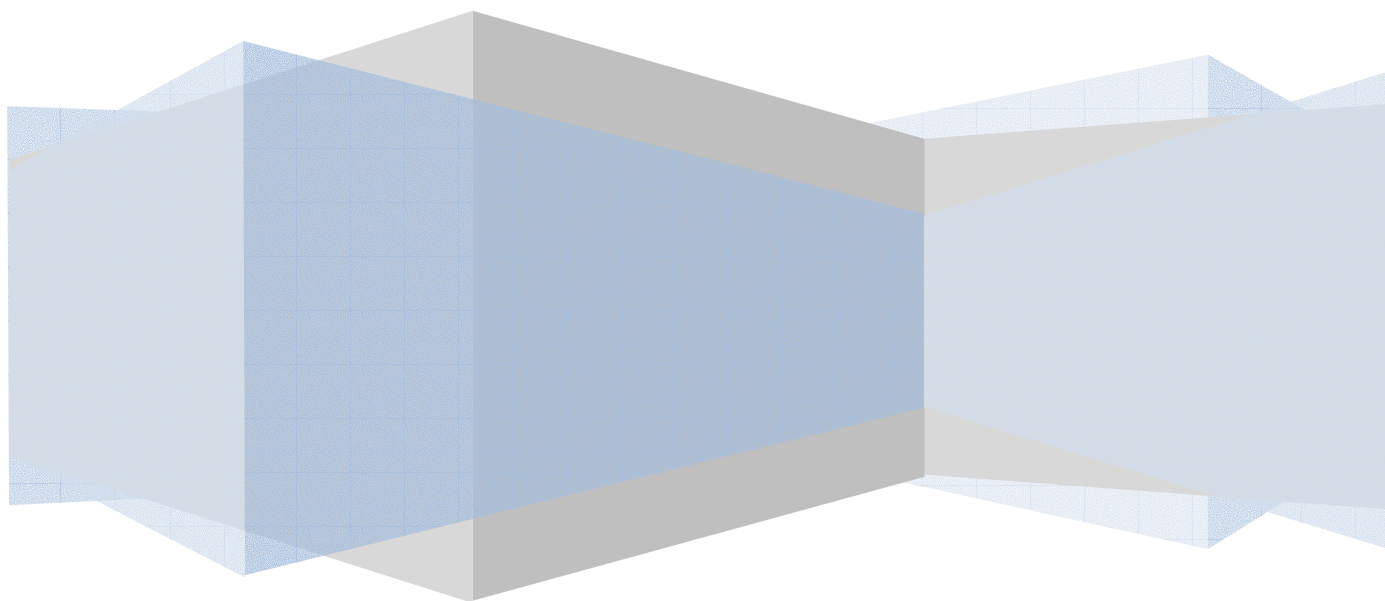
IB Mathematics SL

Specimen Portfolio

Continued Fractions

Type I Tasks

By: Bradian Muliadi



Continued Fractions

A continued fraction is any mathematical expression in the form of:

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6 + \dots}}}}}}$$

Where a_0 is always an integer, and all other 'a's such as a_1 , a_2 , and a_3 are **positive integers**. The number of terms can either be **finite** or **infinite**. A more convenient way to denote continued fractions such as the one above would be to denote it by:
 $N = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, \dots]$

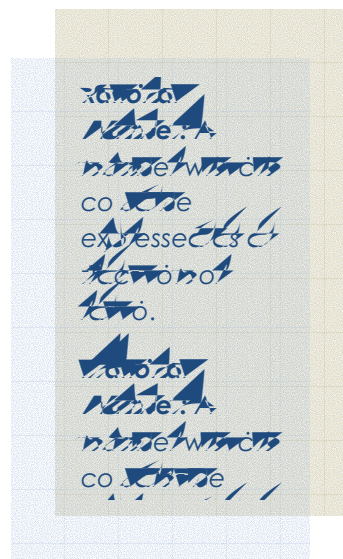
Finite Continued Fractions

A finite continued fraction is an expression such as the one shown above which could end. Every rational number can be equated to a finite continued fraction. The only skill needed would be division of fractions.

$$\begin{aligned} \frac{37}{9} &= 17 + \frac{14}{9} = 17 + \frac{1}{\frac{9}{14}} = 17 + \frac{1}{1 + \frac{5}{14}} = 17 + \frac{1}{1 + \frac{1}{\frac{14}{5}}} \\ &= 17 + \frac{1}{1 + \frac{1}{2 + \frac{4}{5}}} = 17 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{5}{4}}}} = 17 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4}}}} \\ &= [17, 1, 2, 1, 4] \end{aligned}$$

Infinite Continued Fractions

Unlike the finite continued fractions, the chain of fractions never ends in an infinite continued fraction. Every irrational number can be equated to an infinite continued



fraction. This fact was discovered and proven by the Swiss Mathematician, Leonhard Euler (1707-1783). Some of Euler's infinite continued fractions are as we will see below:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

A way to summarise this expression is to let x denote the value of the continued fraction.

$$x = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}} \Rightarrow x = 1 + \frac{1}{1 + x}$$

Usage of Continued Fractions

Continued fractions could be used to solve certain quadratic equations of the second degree. Solving a quadratic equation using the 'completing the square' method only gives the roots of the equation, on the other hand solutions are often required to be expressed as a fraction or decimal fraction. That is why

$x^2 = 2$ continued fractions are used to solve quadratic irrational numbers.

$$x^2 - 1 = 1$$

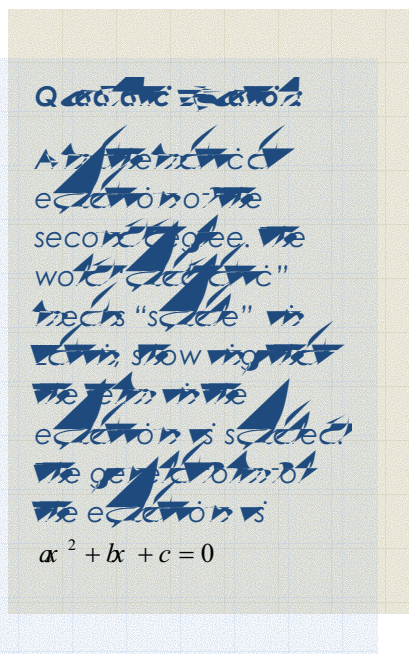
$$(x - 1)(x + 1) = 1$$

$$x - 1 = \frac{1}{1 + x}$$

$$x = 1 + \frac{1}{1 + (1 + \frac{1}{1 + x})} = 1 + \frac{1}{2 + \frac{1}{1 + x}}$$

$$x = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} = \sqrt{2}$$

Example:



Questions A

1. Let n be 2

$$t_n = 1 + \frac{1}{1+1}$$

$$t_{n+1} = 1 + \frac{1}{t_n} = 1 + \frac{1}{1 + \frac{1}{1+1}} \quad \left. \vphantom{t_{n+1}} \right\} = t_3$$

Let n be 3

$$t_n = 1 + \frac{1}{1 + \frac{1}{1+1}}$$

$$t_{n+1} = 1 + \frac{1}{t_n} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}} \quad \left. \vphantom{t_{n+1}} \right\} = t_4$$

Let n be 4

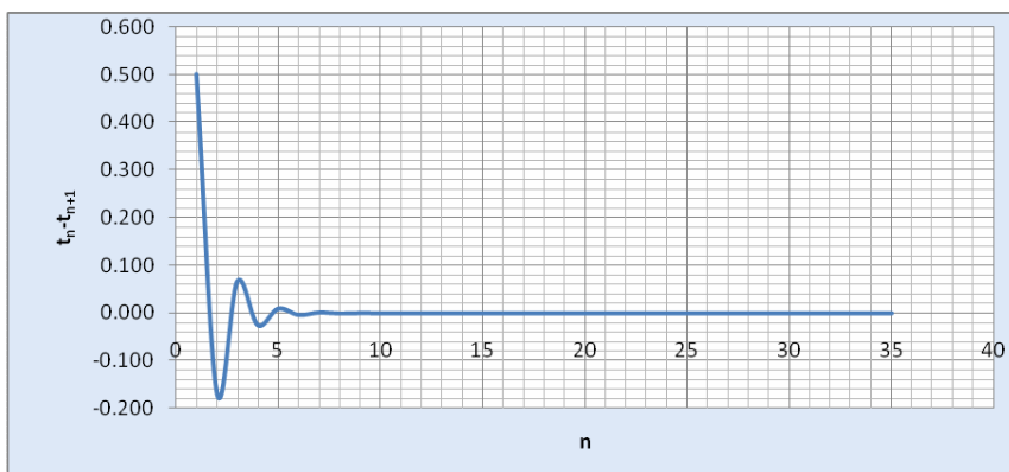
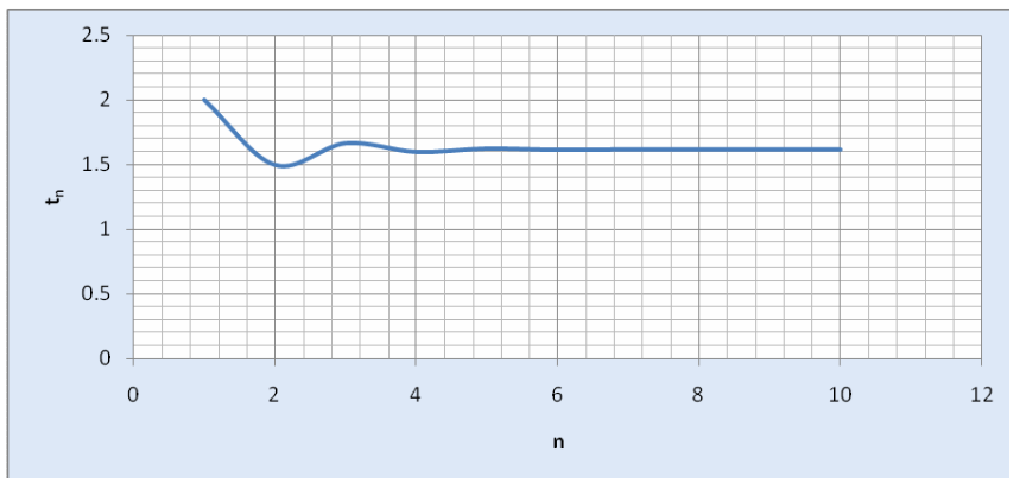
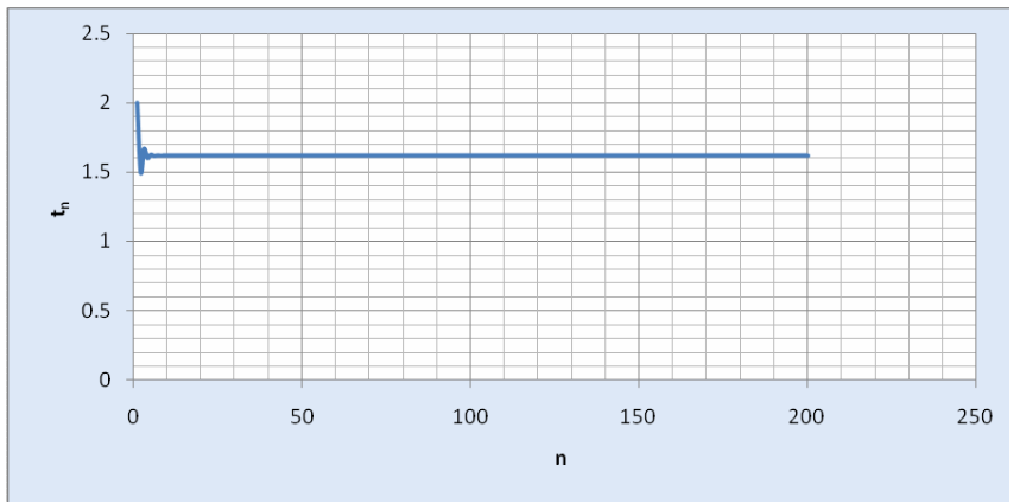
$$t_n = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}$$

$$t_{n+1} = 1 + \frac{1}{t_n} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}} \quad \left. \vphantom{t_{n+1}} \right\} = t_5$$

Conclusion: Therefore $t_{n+1} = 1 + \frac{1}{t_n}$ whereby $n > 1$

2.

n	t_n	t_{n+1}	$t_n - t_{n+1}$
1	2	1.5	0.5000000000
2	1.5	1.666666667	-0.1666666667
3	1.666666667	1.6	0.0666666667
4	1.6	1.625	-0.0250000000
5	1.625	1.615384615	0.0096153846
6	1.615384615	1.619047619	-0.0036630037
7	1.619047619	1.617647059	0.0014005602
8	1.617647059	1.618181818	-0.0005347594
9	1.618181818	1.617977528	0.0002042901
10	1.617977528	1.618055556	-0.0000780275
11	1.618055556	1.618025751	0.0000298045
12	1.618025751	1.618037135	-0.0000113842
13	1.618037135	1.618032787	0.0000043484
14	1.618032787	1.618034448	-0.0000016609
15	1.618034448	1.618033813	0.0000006344
16	1.618033813	1.618034056	-0.0000002423
17	1.618034056	1.618033963	0.0000000926
18	1.618033963	1.618033999	-0.0000000354
19	1.618033999	1.618033985	0.0000000135
20	1.618033985	1.61803399	-0.0000000052
21	1.61803399	1.618033988	0.0000000020
22	1.618033988	1.618033989	-0.0000000008
23	1.618033989	1.618033989	0.0000000003
24	1.618033989	1.618033989	-0.0000000001
25	1.618033989	1.618033989	0.0000000000
26	1.618033989	1.618033989	0.0000000000
27	1.618033989	1.618033989	0.0000000000
28	1.618033989	1.618033989	0.0000000000
29	1.618033989	1.618033989	0.0000000000
30	1.618033989	1.618033989	0.0000000000
31	1.618033989	1.618033989	0.0000000000
32	1.618033989	1.618033989	0.0000000000
33	1.618033989	1.618033989	0.0000000000
34	1.618033989	1.618033989	0.0000000000
35	1.618033989	1.618033989	0.0000000000



▲ According to the graphs, as the value of n increases, t_n will continuously fluctuate but start to stabilize when $n=8$. Once n reaches 23, t_n will converge to a sustained value of 1.618033989. This value will remain constant as long as $n \geq 23$

The same trend is brought up by the graph of n and $t_n - t_{n+1}$. ▲ As the value of n rises, $t_n - t_{n+1}$ will oscillate until n reaches 25. Only then will the value of $t_n - t_{n+1}$ converge to a consistent value of 0. This value will stay the same as long as $n \geq 25$.

3. When trying to determine the 200th term, we couldn't obtain a negative value, only a positive value. When the value of n is still below 25, it oscillates between values which sometimes range from negative to positive values, but then as the value of n increases it will converge to a constant specific positive value. That means a negative value could only be obtained if $n < 25$
4. ▲ According to the table in question two, the exact value for the continued fraction is 1.618033989 because as the value of n increases, the continued fraction's value converges into 1.618033989 and it remains constant as long as $n \geq 23$.

Questions B

1. Let n be 2

$$t_n = 2 + \frac{1}{2+1}$$

$$t_{n+1} = 2 + \frac{1}{t_n} = 2 + \frac{1}{2 + \frac{1}{2+1}} \quad \left. \vphantom{t_{n+1}} \right\} = t_3$$

Let n be 3

$$t_n = 2 + \frac{1}{2 + \frac{1}{2+1}}$$

$$t_{n+1} = 2 + \frac{1}{t_n} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2+1}}} \quad \left. \vphantom{t_{n+1}} \right\} = t_4$$

Let n be 4

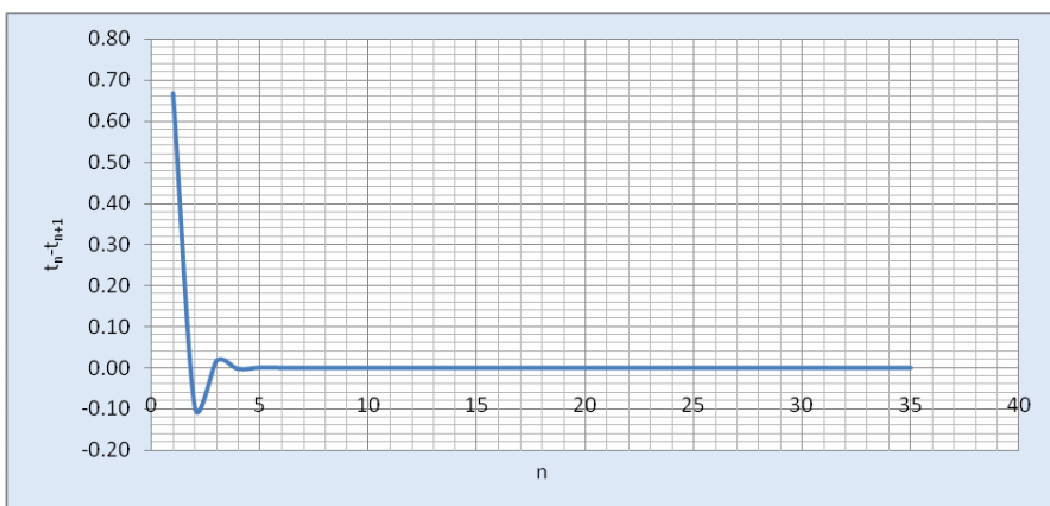
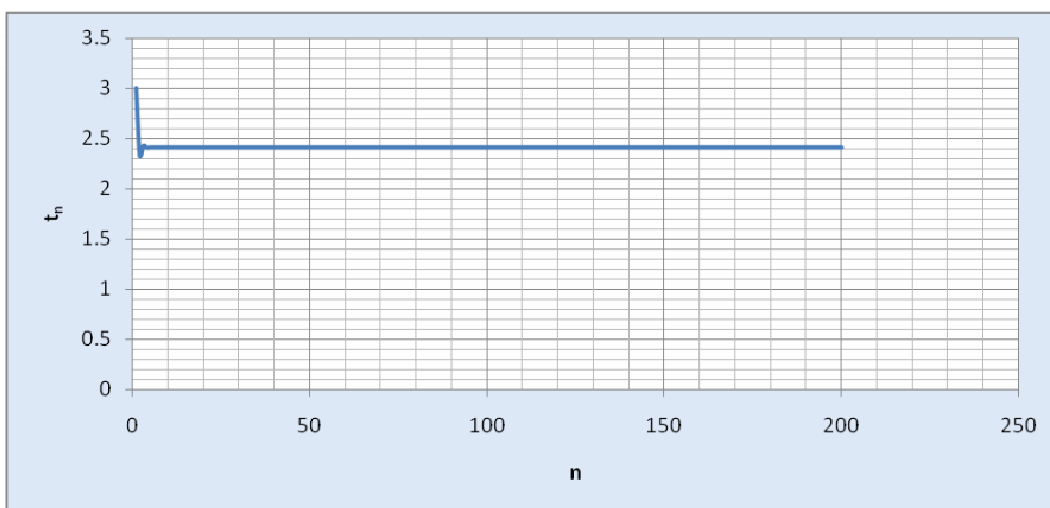
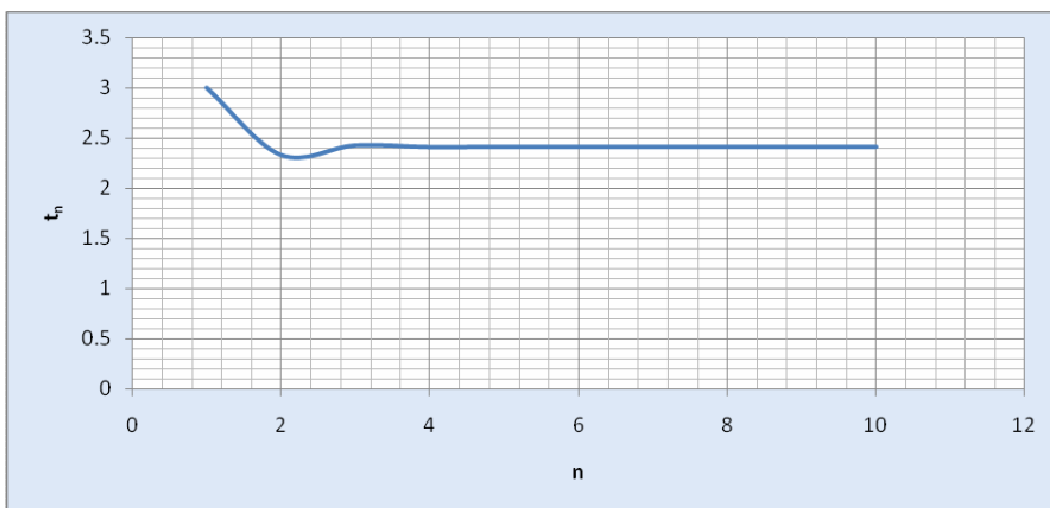
$$t_n = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2+1}}}$$

$$t_{n+1} = 2 + \frac{1}{t_n} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2+1}}}} \quad \left. \vphantom{t_{n+1}} \right\} = t_5$$

Conclusion: Therefore $t_{n+1} = 2 + \frac{1}{t_n}$ whereby $n > 1$

2.

n	t_n	t_{n+1}	$t_n - t_{n+1}$
1	3	2.333333333	0.666666667
2	2.333333333	2.428571429	-0.0952380952
3	2.428571429	2.411764706	0.0168067227
4	2.411764706	2.414634146	-0.0028694405
5	2.414634146	2.414141414	0.0004927322
6	2.414141414	2.414225941	-0.0000845273
7	2.414225941	2.414211438	0.0000145029
8	2.414211438	2.414213927	-0.0000024883
9	2.414213927	2.4142135	0.0000004269
10	2.4142135	2.414213573	-0.0000000732
11	2.414213573	2.414213561	0.0000000126
12	2.414213561	2.414213563	-0.0000000022
13	2.414213563	2.414213562	0.0000000004
14	2.414213562	2.414213562	-0.0000000001
15	2.414213562	2.414213562	0.0000000000
16	2.414213562	2.414213562	0.0000000000
17	2.414213562	2.414213562	0.0000000000
18	2.414213562	2.414213562	0.0000000000
19	2.414213562	2.414213562	0.0000000000
20	2.414213562	2.414213562	0.0000000000
21	2.414213562	2.414213562	0.0000000000
22	2.414213562	2.414213562	0.0000000000
23	2.414213562	2.414213562	0.0000000000
24	2.414213562	2.414213562	0.0000000000
25	2.414213562	2.414213562	0.0000000000
26	2.414213562	2.414213562	0.0000000000
27	2.414213562	2.414213562	0.0000000000
28	2.414213562	2.414213562	0.0000000000
29	2.414213562	2.414213562	0.0000000000
30	2.414213562	2.414213562	0.0000000000
31	2.414213562	2.414213562	0.0000000000
32	2.414213562	2.414213562	0.0000000000
33	2.414213562	2.414213562	0.0000000000
34	2.414213562	2.414213562	0.0000000000
35	2.414213562	2.414213562	0.0000000000



In line with the graphs, as the value of n increases, t_n will persistently oscillate but start to stabilize when $n=4$. When n is 14, t_n will converge to an exact value of 2.414213562. This value will not change as long as $n \geq 14$

The graph of n and $t_n - t_{n+1}$ follows the same trend in that as the value of n ascends, $t_n - t_{n+1}$ will oscillate until n reaches 15. Only then will the value of $t_n - t_{n+1}$ converge to a consistent value of 0. This value will stay the same as long as $n \geq 15$.

3. When trying to determine the 200th term, we couldn't obtain a negative value, only a positive value. When the value of N is still below 15, it oscillates between values which sometimes range from negative to positive values, but then as the value of N increases it will converge to a constant specific positive value. In that case, N could only be a negative value if $n < 15$
4. Pursuant to the table in question two, the exact value for the continued fraction is 2.414213562 because as the value of n increases, the continued fraction's value converges into 2.414213562 and it remains constant as long as $n \geq 14$.

Considering the general continued fraction:

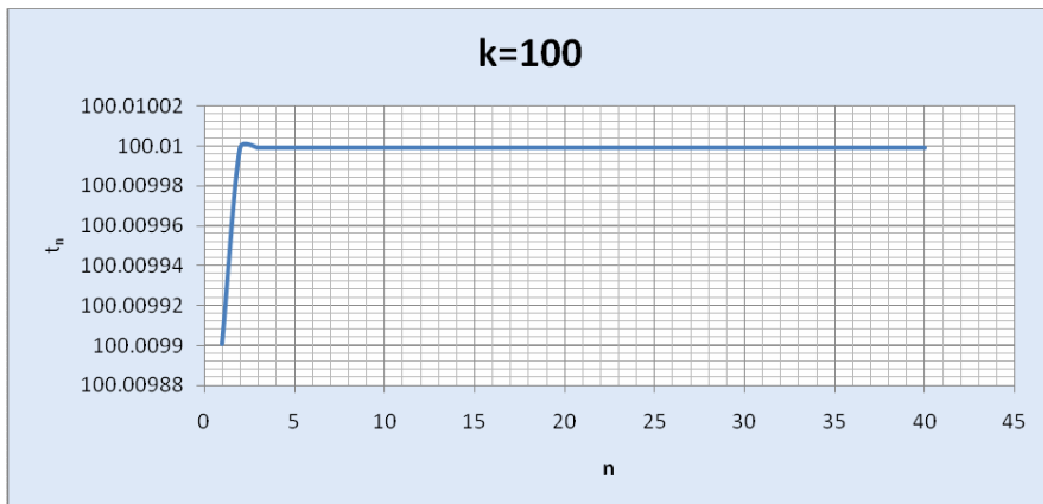
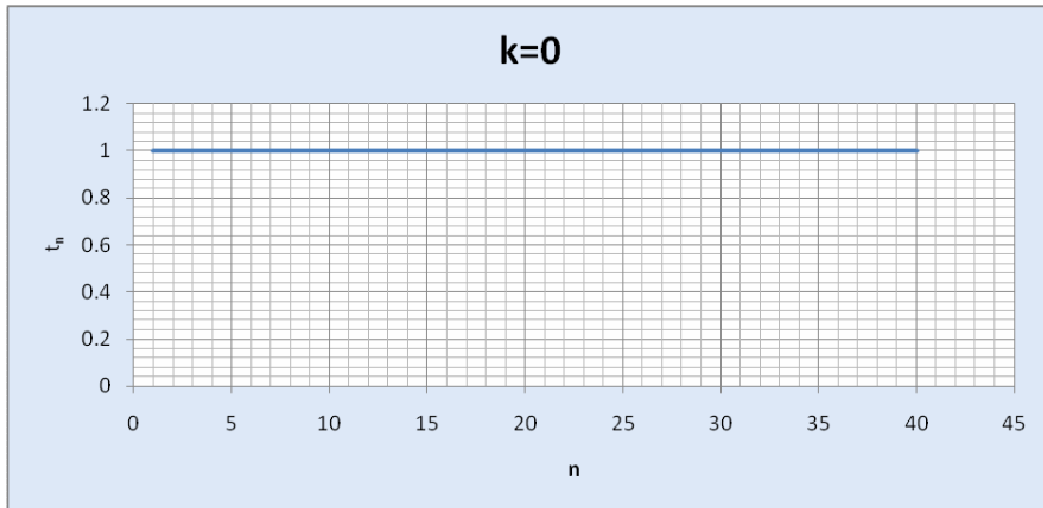
$$k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{\dots}}}}}}}$$

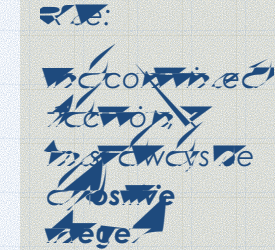
Determine a generalized statement for the exact value of any such continued fraction. For which values of k does your generalized statement hold true? How do you know? Provide evidence to support your answer.

The exact value of a continued fraction as shown above could be determined when plotting a graph. In the graph, the line will initially oscillate above and below the exact value and then finally converge to it. The point where it converges and remains constant is the exact value of the continued fraction.

This statement only holds true if $k \geq 0$ and is an integer. Observe the table below.

n	t_n when $k=0$	t_n when $k=100$	t_n when $k=200$
1	1	100.009901	200.0049751
2	1	100.009999	200.0049999
3	1	100.009999	200.0049999
4	1	100.009999	200.0049999
5	1	100.009999	200.0049999
6	1	100.009999	200.0049999
7	1	100.009999	200.0049999
8	1	100.009999	200.0049999
9	1	100.009999	200.0049999
10	1	100.009999	200.0049999
11	1	100.009999	200.0049999
12	1	100.009999	200.0049999
13	1	100.009999	200.0049999
14	1	100.009999	200.0049999
15	1	100.009999	200.0049999
16	1	100.009999	200.0049999
17	1	100.009999	200.0049999
18	1	100.009999	200.0049999
19	1	100.009999	200.0049999
20	1	100.009999	200.0049999
21	1	100.009999	200.0049999
22	1	100.009999	200.0049999
23	1	100.009999	200.0049999
24	1	100.009999	200.0049999
25	1	100.009999	200.0049999
26	1	100.009999	200.0049999
27	1	100.009999	200.0049999
28	1	100.009999	200.0049999
29	1	100.009999	200.0049999
30	1	100.009999	200.0049999
31	1	100.009999	200.0049999
32	1	100.009999	200.0049999
33	1	100.009999	200.0049999
34	1	100.009999	200.0049999
35	1	100.009999	200.0049999





graphs oscillate or not, a value. If ϵ were to be a math error occurs and a reaction; if it weren't an active ϵ , it will no longer be

http://images.google.com.sg/imgres?imgurl=http://upload.wikimedia.org/math/a/b/d/abda6e157fb26b5fb84d43df36a8ad31.png&imgrefurl=http://en.wikipedia.org/wiki/Solving_quadratic_equations_with_continued_fractions&h=169&w=296&sz=2&hl=en&start=14&um=1&usq= bg1UGcYe dOsasg8Zdzf-Vawfjk=&tbnid=qfG0CexZok9QDM:&tbnh=66&tbnw=116&prev=/images%3Fq%3Dcontinued%2Binfinite%2Bfraction%26um%3D1%26hl%3Den%26sa%3DG

END OF PORTFOLIO