Pure mathematics is, in its way, the poetry of logical ideas. ~Albert Einstein

Binomial Coefficient

Investigation

Tran Quoc Hoang Viet

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Math Portfolio

- Pascal Triangle is one of the most intersting way to place number in a logical pattern.This is done through very simple steps. Firstly, we start with the number 1 placed at the top.We then write down the following number on the lone below it,trying to form an imaginary triangle form. Each muber is calculated from the addition of the two numbers above it. The exceptions are the number "1" which are near the edge.
- In this portfolio, I am going to find out and prove various formula and sums that are interconnected with each other through a common element: The Pascal Triangle. For certain time, I will have to prove certain symmetrical property, to find the sums, to work out the general formula and prove it is true, taking true examples of it. Explicating some of the formula connecting with the Pascal Triangle would be what I am about to do for most of my portfolio

1

Row 1

1 1

Row 1

1 2 1

Row 2

1 3 3 1

Row 3

1 4 6 4 1

Row 4

1 5 10 10 5 1

Row 5

1 6 15 20 15 6 1

Row 6

1 7 21 35 35 21 7 1

Row 7

1 8 28 56 70 56 28 8 1

Row 8

1 9 36 84 126 126 84 36 9 1

Row 9

1 10 45 120 210 252 210 120 45 10 1

Row 10

1 11 55 165 330 462 462 330 165 55 11 1

Row 11

1 12 66 220 495 792 924 792 495 220 66 12 1

Row 12

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1 Row13

The binomial coefficients in the expansion of $(a+b)^n, n \in z^+$ are defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, n \geq r(= nCr)$$

We have the equation of $\binom{n}{r}=\binom{n}{n-r}$ which is further derived from the above Triangle. I am now going to prove that this formula really work as true formula. Having prove this formula, will be able to get the subsequent number for a certain row

For the right hand side, we can expand it further by substituting the value of r as n-r

$$\binom{n}{n-r}=\frac{n!}{(n-r)!(n-n-r)!}$$

$$\binom{n}{n-r}=\frac{n!}{r!(n-r)!}$$

At this point, we are able to see that:

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}=\binom{n}{n-r}$$ (Proved)

I now start to put in random number in order to test the mentioned formula out. My chosen number here is 5 and my k is 2.

$\binom{5}{2}=\binom{5}{3}$ (True)

This result proves that $\binom{n}{k}=\binom{n}{n-r}$

Here we have a few sums. I will have to use my drawn Pascal Triangle in order to calculate all of these. Most of the given coefficients are within the range of number of rows in my drawn Pascal Triangle.

$\binom{5}{3}+\binom{5}{4}$ = 10 +7 =17

$\binom{8}{2}+2\binom{8}{3}+\binom{8}{4}$ =28+ + 2.56 + 70 = 28 + 112 +70 =210

$\binom{9}{4}+3\binom{9}{5}+3\binom{9}{6}+\binom{9}{7}$ =126 + 3.126 + 3.84 + 36 = 126 + 378 +252 + 36 =792

$\binom{7}{2}+4\binom{7}{3}+6\binom{7}{4}+4\binom{7}{5}+\binom{7}{6}$ =7+4.21+4.35+6.35+4.21+7=7+140+210+84+7=462

Using all of these sums, we can figure out a general formula for each type of sum. I am going to use Algebra as the main method of proving all of the general formulas for these sums as true formulas.

$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$ is the general formula for the first sum: $\binom{5}{3}+\binom{5}{4}$

On the left hand side, we have to substitute r as r-1 and $\binom{n}{r}$ as $\frac{n!}{r!(n-r)!}$

$$=\frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}$$

We can further simplify r! by these steps: r! = r(r-1)(r-2)….

$$\frac{r!}{r} = (r-1)!$$

I now substitute this answer into the above line

$$=\frac{n!}{\frac{r!}{r}(n-r+1)!}+\frac{n!}{r!(n-r)!}$$

▲ fraction within a fraction, the nominator can be moved to the upper nominator

$$=\frac{n!r}{r!(n-r+1)!}+\frac{n!}{r!(n-r)!}$$

(n-r+1)! Can be further simplified through these steps: (n-r+1)! = (n-r+1)(n-r)!

$$(n-r)! = \frac{(n-r+1)!}{n-r+1}$$

$$=\frac{n!r}{r!(n-r+1)!}+\frac{n!}{\frac{(n-r+1)!}{n-r+1}}$$

▲ fraction within a fraction, the nominator can be moved to the upper nominator

$$=\frac{n!r}{r!(n-r+1)!}+\frac{n(n-r+1)!}{r!(n-r+1)!}$$

Both fraction has the same denominator, they can join each other

$$=\frac{n!r+n!(n-r+1)}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$= \binom{n+1}{r}$ (as in the required form)

Hence we can conclude that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Again, a general formula for the second sum, letter b, can be produced from the sum.That general formula is $\binom{n}{r-2} + 2\binom{n}{r-2} + \binom{n}{r}$. I am now going to use the proved formula and algebra rules to prove that the formula is the true formula for it.

I split $2\binom{n}{r-2}$ into 2 singles: $\binom{n}{r-1}$

$$= \binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r-1} + \binom{n}{r}$$

Using the proved formula, $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$

$$= \binom{n+1}{r-1} + \binom{n+1}{r}$$

Using the proved formula, $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$

$$= \binom{n+2}{r}$$

Lastly, the third formula for the third sum, which is letter c, is $\binom{n}{r-3} + 3\binom{n}{r-2} + 3\binom{n}{r-1} + \binom{n}{r}$. Similarly, I am going to use both of the proved formulas in order to prove this formula

I split $3\binom{n}{r-2}$ into $2\binom{n}{r-2}$ and $\binom{n}{r-2}$

I split $3\binom{n}{r-1}$ into $2\binom{n}{r-1}$ and $\binom{n}{r-1}$

$$= \binom{n}{r-3} + 2\binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$$

Using the proved formula $\binom{n}{r-2} + 2\binom{n}{r-2} + \binom{n}{r}$

$= \binom{n+2}{r-1} + \binom{n+2}{r}$

Using the proved formula $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$= \binom{n+3}{r}$

According to the Pascal Triangle, there is a formula connecting any (k+1) successive coefficients. We will have to find that formula which connects the mentioned coefficients in the nth row of the Pascal triangle with a coefficient in the $(n+k)^{Th}$ row.

$$(n+k)^r = \sum_{k=0}^{r} \binom{n}{r}\binom{n}{r-k}$$

$$\binom{n+k}{r} = \binom{n}{r-k}\binom{k}{0} + \binom{k}{r-k-1}\binom{k}{1} + \binom{n}{r-k-2}\binom{k}{2} + \cdots \binom{n}{r}\binom{k}{k}$$

$$\binom{n+k}{r} = \sum_{i=0}^{k} \binom{k}{i}\binom{n}{r-i}$$

I am going to use the mathematical induction in order to prove it is true.

Firstly, for the first P_0, we can see it quite straightforward

$$\binom{n}{r} = \sum_{i=0}^{0} \binom{0}{i}\binom{n}{r-0}$$

$$= 1\binom{n}{r}$$

$$= \binom{n}{r} \text{ (True)}$$

For P_k , we want to show that the substituted formula is true

$$\binom{n+k}{r} = \sum_{i=1}^{k} \binom{k}{1}\binom{n}{r-1}$$

(1)

Assuming (1) is true, we want to prove, that it's also true for P_{k+1}.We now start to substitute the k as k+1 for the whole equation.

$$\binom{n+k+1}{r} = \sum_{i=0}^{k+1} \left[\binom{k}{i-1} + \binom{k}{1}\right]\binom{n}{r-1}$$

Using the algebra rule, I expand from inside the brackets and get the answers.

$$= \sum_{i=0}^{k+1} \binom{k}{i-1}\binom{n}{r-i} + \sum_{i=0}^{k+1} \binom{k}{i}\binom{n}{r-i}$$

$$\sum_{i=0}^{k+1} \binom{k}{i}\binom{n}{r-i}$$

We can split this big sum into 2 elements in order to get the desired elements.

$$= \sum_{i=0}^{k+1} \binom{k}{i-1}\binom{n}{r-i} + \sum_{i=0}^{k} \binom{k}{i}\binom{n}{r-i} + \binom{k}{k+1}\binom{n}{r-k-1}$$

We should be able to arrive at the equation of $\binom{n+k}{r}$ according to the stated assumption. The newly created factor is 0 due to the fact that the equation is not included inside the Pascal Triangle

$$= \sum_{i=0}^{k+1} \binom{k}{i-1}\binom{n}{r-i} + \binom{n+k}{r} + 0$$

Similar to the splitting steps, we again simplify the complex equation into 2 other elements.

$$= \binom{k}{-1}\binom{n}{r} + \sum_{i=1}^{k} \binom{k}{i-1}\binom{n}{r-i} + \binom{n+k}{r}$$

The new factor also cannot be found in the Pascal Triangle and is considered as 0.Having the last 2 elements, we can use the proved formula of $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$ to arrive at the final answer.

$$= 0 + \binom{n+k}{r-1} + \binom{n+k}{r}$$

$$= \binom{n+k+1}{r}$$ (As in the required form)

Conclusion: P_0 is true and P_{k+1} is true whenever P_k is true $\forall$ n □ N^*.

I am now going to show that the formula really works for the real values once I start to substitute them in. In this case, the sum that I have to do is

$$\sum_{i=0}^{8} \binom{8}{r}\binom{27}{8-r}$$

Normally, I would solve this problem through the binomial expansion using the proved formula in previous part:

$$\sum_{r=0}^{8} \binom{8}{r}\binom{27}{8-r} = \binom{8}{0}\binom{27}{8} + \binom{8}{1}\binom{27}{7} + \binom{8}{2}\binom{27}{6} + \binom{8}{3}\binom{27}{5} + \binom{8}{4}\binom{27}{4} + \binom{8}{5}\binom{27}{3} + \binom{8}{6}\binom{27}{2} + \binom{8}{7}\binom{27}{1} + \binom{8}{8}\binom{27}{0}$$

$= 2{,}220{,}075$ + 7,104,240 + 8,288,280 + 4,520,880 + 1,228,500 + 163,800 + 9,828 + 216 + 1

$= 23{,}535{,}440$

There is also another way to solve this sum, using the original binomial coefficient

$$\sum_{i=0}^{8} \binom{8}{r}\binom{27}{8-r} = \binom{27+8}{8} = \binom{35}{8} = \frac{35!}{8!\,(35-8)!} = 23{,}535{,}820$$

Through getting the 2 same results, we can see that the formula really work no matter what the method. Thus we can come to a conclusion that the formula works and it is true.

Beside the formula derived further from the Pascal Triangle, a formula exists for calculating The sum of the squares of the elements of row n equals the middle element of row (2n − 1).That thing is represented through a general formula:

$$\sum_{r=0}^{n}\binom{n}{r}^2$$

I will now have to find the final answer for this sum using the proved formulas in the other parts. At the same time I am going to use algebraic rule to simplify the elements in order to arrive at the final answer.

$$\sum_{r=0}^{n}\binom{n}{r}^2 = \sum_{r=0}^{n}\binom{n}{r}\binom{n}{r}$$

Using the proved formula of $\binom{n}{r} = \binom{n}{n-r}$, I am able to manipulate the equation into another form.

$$= \sum_{r=0}^{n}\binom{n}{r}\binom{n}{n-r}$$

Again, using the proved formula , I will be able to get the final answer for the sum

$$\binom{n+k}{r} = \sum_{i=1}^{k}\binom{k}{1}\binom{n}{r-1}$$

$$= \sum_{r=0}^{n}\binom{n+n}{n}$$

$$= \sum_{r=0}^{n}\binom{2n}{n}$$

Once I get to the final answer, I now have to prove that the formula works for all cases with the help of real numbers replacing inside the formula itself. The sum that I have to solve as an example for the working formula is:

$$\sum_{r=0}^{13}\binom{13}{r}^2$$

Firstly, I would start to solve this sum through normal binomial expansion. Taking the number from the drawn Pascal Triangle, we can see

$$\sum_{r=0}^{13}\binom{13}{r}^2 = \binom{13}{0}^2 + \binom{13}{1}^2 + \binom{13}{2}^2 + \binom{13}{3}^2 + \binom{13}{4}^2 + \binom{13}{5}^2 + \binom{13}{6}^2 + \binom{13}{7}^2 + \binom{13}{8}^2 + \binom{13}{9}^2 + \binom{13}{10}^2 + \binom{13}{11}^2 + \binom{13}{12}^2 + \binom{13}{13}^3$$

$$= 1 + 169 + 6{,}084 + 81{,}796 + 511{,}225 + 1{,}656{,}369 + 2{,}944{,}656 + 2{,}944{,}656 + 1{,}656{,}369 + 511{,}225 + 81{,}796 + 6{,}084 + 169 + 1$$

$$= 10{,}400{,}600$$

Another way of doing this sum is to use the binomial coefficient formula to go about doing this sum.

$$\sum_{r=0}^{13}\binom{13}{r}^2 = \binom{26}{13} = \frac{26!}{13!\,(26-13)!} = \frac{26!}{13!^2} = 10{,}400{,}600$$

Getting the same results from two different methods, this tells us that the formula really works on different cases. We are then concludes that the formula is true and working.

In the conclusion words, I would like to summarize what I have done so far in this portfolio. Firstly, I got to prove the symmetrical property for the coefficients, which is $\binom{n}{r} = \binom{n}{n-r}$. Thus, I am able to conduce to another general formula from the sums that I was given to do. These are the three formulas:

$$\binom{n}{r-1}+\binom{n}{r} \qquad \binom{n}{r-2} + 2\binom{n}{r-2}+\binom{n}{r} \qquad \binom{n}{r-3} + 3\binom{n}{r-2} + 3\binom{n}{r-1} + \binom{n}{r}$$

and the sums:

$$\binom{5}{3}+\binom{5}{4} \qquad \binom{8}{2}+2\binom{8}{3}+\binom{8}{4} \qquad \binom{9}{4}+3\binom{9}{5}+3\binom{9}{6}+\binom{9}{7}$$

Once I got my hands on the answer for the sums, I then proceeded onto find the formula which connect any (k +1) successive coefficients in the nth row of the Pascal Triangle with the coefficient in the $(n+k)^{th}$ row. Should I am able to find it; I then, with the use of algebraic rule and mathematical induction, prove the formula is true. Besides using the general elements, I directly test the formulas through real number examples. Here are the two formulas that I managed to get:

$$\binom{n+k}{r} = \sum_{i=1}^{k}\binom{k}{1}\binom{n}{r-1} \qquad \sum_{r=0}^{n}\binom{n}{r}^2 = \sum_{r=0}^{n}\binom{2n}{r}$$

Consequently, I have to use real number sums to test these out. These are the sums that I did:

$$\sum_{r=0}^{13}\binom{13}{r}^2 \qquad \sum_{i=0}^{8}\binom{8}{r}\binom{27}{8-r}$$

The Pascal Triangle is one in a million mathematical wonders that human can ever think of. It connects all common senses of logic and algebra rules. From it, various general formulas can be derived from, making more mathematical properties at the same make them easier to understand. More formulas means that there will be less work of calculations required, making the job of a mathematician easier. Although it is not just Blaise Pascal that got to think of such triangle, in the years of the 13th century, an arithmetic triangle which looks and works exactly in the same way as the Western Pascal triangle was produced by a Chinese mathematician .The only slight difference is that it was expressed through Chinese figures.