## LACSAP'S FRACTION

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## Internal Assessment

## IB Mathematics SL

## Portfolio Type 1

When we look at the Lacsap's fraction, probably the first thing we notice is a similarity in the shape of a pyramid of Pascal's triangle (as shown in Figures 1 and 2).

$$
\begin{array}{ccccccc} 
& & & & 1 & & \\
& & & 1 & 1 & & \\
& & & & & & \\
& & & \frac{3}{2} & & & \\
& & & & & \\
& & & \frac{6}{4} & & & \frac{6}{4} \\
& & 1 & & \\
& & \frac{10}{7} & & \frac{10}{6} & & \frac{10}{7} \\
& & & 1 & \\
\frac{15}{11} & & \frac{15}{9} & & \frac{15}{9} & & \frac{15}{11} \\
& & 1
\end{array}
$$

Figure 1. Pattern of Lacsap's fractions

$$
\left.\begin{array}{cccccccc} 
& & & & 1 & & & \\
& & & & & & & \\
& & & 1 & 2 & 1 & & \\
& & 1 & 3 & & 3 & & 1
\end{array}\right)
$$

Figure 2. Pascal's triangle

However, by paying more attention to these two patterns, we will notice some dependence.
n - number of row

$$
\begin{aligned}
& \mathrm{n}=1 \\
& \mathrm{n}=2 \\
& 11 \\
& \frac{3}{2} \\
& 1 \quad \overline{2} \quad 1 \\
& \mathrm{n}=3 \\
& 1 \quad \frac{6}{4} \quad \frac{6}{4} \quad 1 \\
& \mathrm{n}=4 \\
& 1 \quad \frac{10}{7} \quad \frac{10}{6} \quad \frac{10}{7} \quad 1 \\
& \mathrm{n}=5 \\
& 1 \quad \frac{15}{11} \quad \frac{15}{9} \quad \frac{15}{9} \quad \frac{15}{11} \quad 1
\end{aligned}
$$

$\mathrm{n}=0$
$\mathrm{n}=1$
$\mathrm{n}=2$
$n=3$
1331
$\mathrm{n}=4$
14
$6 \quad 4 \quad 1$
$\mathrm{n}=5$
$\begin{array}{llllll}1 & 5 & 10 & 10 & 5 & 1\end{array}$

The property mentioned above shows that the numbering of each row in the Lacsap's pattern (starting with $n=1$ ) is equal to the third element of the next row in Pascal's triangle. We can define the following dependence by:

$$
{ }_{(n+1)} C_{2}=m
$$

where:
$m$ - numerator
$n$ - row number

To check whether this equation is correct, we can go through a series of sample calculations:

For $n=2$
$(2+1) \mathrm{nCr} 2$
3

For $n=3$
$(3+1) \mathrm{nCr} 2$
6

For $n=4$
(4+1) $\mathrm{nCr}^{2}$

For $n=5$

## $(5+1) \mathrm{nCr} 2$

## 15

When we checked whether the calculations are correct, we can then calculate the numerator of $6^{\text {th }}$ and $7^{\text {th }}$ row by taking the same operations:

For $n=6$
$(6+1) \mathrm{nCr} 2$

## 21

For $n=7$
$(7+1) \mathrm{nCr} 2$
28

Afterwards we have to complete our calculations by finding the dominator. There is also a trend for this:

$$
\mathrm{r}=1
$$

$$
1+0 \quad 1+0
$$

$$
1+0 \quad{ }^{\frac{3}{2}}+1 \quad 1+0
$$

$$
\mathrm{r}=3
$$

$$
1+0 \quad \frac{6}{4}+2 \frac{6}{4}_{+2} \quad 1+0
$$

$\mathrm{r}=4$

$$
1+0 \frac{\frac{10}{7}}{+3} \frac{\frac{10}{6}}{+4} \frac{\frac{10}{7}}{}+3 \quad 1+0
$$

$\mathrm{r}=5$

$$
1+00^{\frac{15}{11}}+4^{\frac{15}{9}}+6^{\frac{15}{9}}+6^{\frac{15}{11}}+41+0
$$

Figure 3. The pattern showing differences between the numerator and dominator in the Lacsap's pattern

The relationship between those differences and the row number can be presented in the table.

| Row <br> number | Difference between <br> numerator and denominator |
| :---: | :---: |
| $\mathbf{1}$ | 0 |
| $\mathbf{2}$ | 1 |
| $\mathbf{3}$ | 2 |
| $\mathbf{4}$ | 3 |
| $\mathbf{5}$ | 4 |

Table 1. Table showing the relationship between the row number and the difference between the numerator and denominator of the $2^{\text {nd }}$ element of each row.

The difference between the numerator and the denominator increases by 1 . Moreover, the difference between the row number and the difference between numeraror and dominator also increases by 1 .

| Row <br> number | Difference between <br> numerator and denominator |
| :---: | :---: |
| $\mathbf{1}$ | $\mathrm{n} / \mathrm{a}$ |
| $\mathbf{2}$ | 0 |
| $\mathbf{3}$ | 2 |


| $\mathbf{4}$ | 4 |
| :---: | :---: |
| $\mathbf{5}$ | 6 |

Table 2. Table showing the relationship between the row number and the difference between the numerator and denominator of the $3^{\text {rd }}$ element of each row.

Since there is $3^{\text {rd }}$ element in the $1^{\text {st }}$ row the difference between the numerator and the denominator does not apply there. In other rows the difference between the numerator and denominator increases by 2 .

| Row <br> number | Difference between <br> numerator and denominator |
| :---: | :---: |
| $\mathbf{1}$ | $\mathrm{n} / \mathrm{a}$ |
| $\mathbf{2}$ | $\mathrm{n} / \mathrm{a}$ |
| $\mathbf{3}$ | 0 |
| $\mathbf{4}$ | 3 |
| $\mathbf{5}$ | 6 |

Table 3. Table showing the relationship between the row number and the difference between the numerator and denominator of the $4^{\text {th }}$ element of each row.

Since there is no no $4^{\text {th }}$ element in the $1^{\text {st }}$ and $2^{\text {nd }}$ row the difference between the numerator and the denominator does not apply there. In other rows the difference between the numerator and the denominator increases by 3

When we have all these patterns recorded in the table, we can deduce certain dependences.
For the $2^{\text {nd }}$ element of each row:

$$
d_{I}=m_{1}-(n-1)
$$

where:
$d_{1}$ - denominator
$m_{l}$ - numerator
$n$ - row number

To check whether this equation is true, we can have some sample calculations:

- for $2^{\text {nd }}$ row: $\mathrm{d}_{1}=3-(2-1)$
$d_{1}=2$
- for $3^{\text {rd }}$ row:

$$
\begin{aligned}
& \mathrm{d}_{1}=6-(3-1) \\
& \mathrm{d}_{1}=4
\end{aligned}
$$

- for $5^{\text {th }}$ row:

$$
\begin{aligned}
& d_{1}=15-(5-1) \\
& d_{1}=11
\end{aligned}
$$

For the $3^{\text {rd }}$ element of each row:

$$
d_{2}=m_{2}-2(n-2)
$$

where:
$d_{2}$ - denominator
$m_{2}$ - numerator
$n$ - row number
Another time, we can check the correctness of the equation:

- for $3^{\text {rd }}$ row:

$$
\mathrm{d}_{2}=6-2(3-2)
$$

$$
\mathrm{d}_{2}=4
$$

- for $4^{\text {th }}$ row:

$$
\begin{aligned}
& \mathrm{d}_{2}=10-2(4-2) \\
& \mathrm{d}_{2}=6
\end{aligned}
$$

- for $5^{\text {th }}$ row:

$$
\begin{aligned}
& \mathrm{d}_{2}=15-2(5-2) \\
& \mathrm{d}_{2}=9
\end{aligned}
$$

For the $3^{\text {rd }}$ element of each row:

$$
d_{3}=m_{3}-3(n-3)
$$

where:
$d_{3}$ - denominator
$m_{3}$ - numerator
$n$ - row number

And again let's make sure that the equation works:

- For the $3^{\text {rd }}$ row:

$$
\begin{aligned}
& \mathrm{d}_{3}=1-3(3-3) \\
& \mathrm{d}_{3}=1
\end{aligned}
$$

- For the $4^{\text {th }}$ row:

$$
\begin{aligned}
& d_{3}=10-3(4-3) \\
& d_{3}=7
\end{aligned}
$$

- For the $5^{\text {th }}$ row:

$$
\begin{aligned}
& d_{3}=15-3(5-3) \\
& d_{3}=9
\end{aligned}
$$

We can notice that each pattern makes sense. From these observations we are able to create a general formula for the denominator, which will be needed later to calculate the dominators of the $6^{\text {th }}$ and $7^{\text {th }}$ raws. The pattern is:

$$
d_{r}=\left(_{(n+1)} C_{2}\right)-r(n-r)
$$

where:
$\mathrm{d}_{\mathrm{r}}$ - denominator
${ }_{(n+1)} \mathrm{C}_{2}$ - numerator
$r$ - number of term
n - row number

With this formula we are now able to determine the dominators in a row 6 and 7 . We must remember that each row is surrounded by a number 1 on both sides and there is no necessity of calculating them. Another important aspect is that Lacsap's pattern is symmetrical, and therefore we have to examine only the first half of the fractions. Thus, in both 6 and 7 rows there will be 3 first fractions needed to calculate (from the Pascal's formula- still excluding ones on the ends).

Calculations for the $6^{\text {th }}$ row:
From our previous experiences that the numerator equals 21.

$$
\begin{aligned}
& (6+1) \mathrm{C}_{2}=\mathrm{m} \\
& { }_{(7)} \mathrm{C}_{2}=21 \\
& (6+1) \text { nCr. } 2
\end{aligned}
$$

21

Hence, for the $1^{\text {st }}$ element (and the $5^{\text {th }}$ ) the dominator equals:

$$
\begin{aligned}
& \mathrm{d}_{1}=21-1(6-1) \\
& \mathrm{d}_{1}=16
\end{aligned}
$$

so the first and fifth term is $\frac{21}{16}$

For $2^{\text {nd }}$ element (and $4^{\text {th }}$ ), the dominator equals:

$$
\begin{aligned}
& \mathrm{d}_{2}=21-2(6-2) \\
& \mathrm{d}_{2}=13
\end{aligned}
$$

so second and fourth term is $\frac{21}{13}$
For the $3^{\text {rd }}$ element in row, the dominator is:

$$
\begin{aligned}
& \mathrm{d}_{3}=21-3(6-3) \\
& \mathrm{d}_{3}=12
\end{aligned}
$$

so the third term is $\frac{21}{12}$
Calculations for the $7^{\text {th }}$ row:
From the previous part of our work we know that the numerator of $7^{\text {th }}$ is 28 .

$$
\begin{aligned}
& (7+1) \mathrm{C}_{2}=\mathrm{m} \\
& { }_{(8)} \mathrm{C}_{2}=28 \\
& (7+1) \text { nCr. } 2
\end{aligned}
$$

28

Hence, for the $1^{\text {st }}$ element (and the $6^{\text {th }}$ ) the dominator equals:

$$
\begin{aligned}
& \mathrm{d}_{1}=28-1(7-1) \\
& \mathrm{d}_{1}=22
\end{aligned}
$$

so the first and fifth term is $\frac{28}{22}$
For $2^{\text {nd }}$ element (and $5^{\text {th }}$ ), the dominator equals:

$$
\begin{aligned}
& \mathrm{d}_{2}=21-2(7-2) \\
& \mathrm{d}_{2}=18
\end{aligned}
$$

so second and fourth term is $\frac{28}{18}$
For the $3^{\text {rd }}$ element in row, the dominator is:

$$
\begin{aligned}
& \mathrm{d}_{3}=21-3(7-3) \\
& \mathrm{d}_{3}=16
\end{aligned}
$$

so the third term is $\frac{28}{16}$

Hence, whole $6^{\text {th }}$ and $7^{\text {th }}$ row will look like these:

$$
\begin{array}{cccccccc}
1 & \frac{21}{16} & \frac{21}{13} & \frac{21}{12} & \frac{21}{13} & \frac{21}{16} & 1 \\
1 & \frac{28}{22} & \frac{28}{18} & \frac{28}{16} & \frac{28}{16} & \frac{28}{18} & \frac{28}{22} & 1
\end{array}
$$

## Conclusion:

With all recorded and processed data, we are able to give a general formula for $\mathrm{r}^{\text {th }}$ element in $\mathrm{n}^{\text {th }}$ row. This is:

$$
E_{n}(r)=\frac{1}{m_{r}} \square E_{n}(r)=\frac{(n+1) C_{2}}{(n+1)} C_{2}-r(n-r)
$$

where:
r - element number
n - row number
This formula is not however ideal, because of several limitations. First of all, it is the number 1 standing at the end of each row that should not be considered in calculating the numerator. Then we have to count the second element of each row as the first one. Secondly, the inequality must be approved $n>0$ otherwise the equation will not work. Finally, the third weakness of this formula is that the very first row of the Lacsap's pattern company is counted as one $1^{\text {st }}$ row.

