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IB Mathematics HL

Portfolio
(Type II)

Arithmetic Sequence Techniques

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1. **Deduce the formula $S_n = \frac{n(a_1 + a_n)}{2}$ for an arithmetic sequence.**

Solution:

Let S = 1 + 2 + ... + 99 + 100

Written another way:
S = 100 + 99 + ... + 2 + 1

Let $a_1 = 1$, $a_2 = 2$, ... , $a_{n-1} = 99$, $a_n = 100$

Notice that 1 + 100 = 2 + 99 = ... = 100 + 1

We can conclude that:
$a_1 + a_n = a_2 + a_{n-1} = \ldots = a_n + a_1$

Let $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$

S_n can also be written as:
$S_n = a_n + a_{n-1} + \ldots + a_2 + a_1$

When we add the two sums together:

$$\begin{aligned}
S_n &= a_1 + a_2 + \quad \ldots \quad + a_{n-1} + a_n \\
+)\ S_n &= a_n + a_{n-1} + \quad \ldots \quad + a_2 + a_1 \\
\hline
2S_n &= (a_1 + a_n) + (a_2 + a_{n-1}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1) \\
&= (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n) \qquad a_1 + a_n = a_2 + a_{n-1} = \ldots = a_n + a_1 \\
&= n(a_1 + a_n) \qquad \textit{n identical terms in the sum}
\end{aligned}$$

$2S_n = n(a_1 + a_n)$

Therefore:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

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2. For an arithmetic sequence, $a_4 + a_{n-3} = 8$ and $S_n = 32$. Find n.

Solution:

From the previous question we know that:
$a_1 + a_n = a_2 + a_{n-1} = ... = a_n + a_1$

So if we extend the formula to:
$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3}... = a_n + a_1$

Then:
$a_4 + a_{n-3} = 8 = a_1 + a_n$

$$S_n = \frac{n(a_1 + a_n)}{2} = 32$$

$$= \frac{n(a_4 + a_{n-3})}{2} = 32$$

$$= \frac{n(8)}{2} = 32$$

$$= 8n = 64$$

Therefore:
n = 8

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3. If $a_1 + a_2 + a_3 = 5$, $a_{n-2} + a_{n-1} + a_n = 10$ and $S_n = 20$, what is n?

Solution:

If we add the two equations together, we get:

$$
\begin{array}{ll}
\quad a_1 + a_2 + a_3 & = 5 \\
+)\ \underline{a_{n-2} + a_{n-1} + a_n} & \underline{= 10} \\
\quad a_1 + a_{n-2} + a_2 + a_{n-1} + a_3 + a_n & = 15 \\
\quad (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) & = 15 \\
\quad 3\,(a_1 + a_n) & = 15 \\
\quad (a_1 + a_n) & = 5
\end{array}
$$

$a_1 + a_n = a_2 + a_{n-1} = \ldots = a_n + a_1$

$$
\begin{aligned}
S_n &= \frac{n\,(a_1 + a_n)}{2} = 20 \\
&= \frac{n\,(5)}{2} = 20 \\
&= 5n = 40
\end{aligned}
$$

Therefore:
n = 8

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4. Using the sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 find the relationship between each pair of results.

I. 2 (7) and (5 + 9); 2(11) and (9 + 13); 2 (13) and (11 + 15)
II. 2 (7 + 9) and (3 + 5) + (11 + 13)
III. 2 (11 + 13 + 15) and (5 + 7 + 9) + (17 + 19 + 21)

For the general arithmetic sequence a_1, a_2, a_3, ... a_{n-1}, a_n, state the general conclusions for I. II. and III.

Then summarize what you have found from I. II. and III.

Solution:

Let $a_1 = 3$, $a_2 = 5$, $a_3 = 7$ ... $a_{10} = 21$

I. $2a_3 = a_2 + a_4$
$2a_5 = a_4 + a_6$
$2a_6 = a_5 + a_7$

So we get:
$2a_n = a_{n-1} + a_{n+1}$

Proof:

From the general arithmetic sequence formula:
$a_n = a_1 + (n - 1)d$:

We know:
$2a_n = 2\ [a_1 + (n - 1)d]$
$a_{n-1} = a_1 + (n – 1 - 1)d]$
$a_{n+1} = a_1 + (n + 1 – 1)d]$

$$\begin{aligned} a_{n-1} + a_{n+1} &= 2a_1 + (2n – 2)d \\ &= 2\ [a_1 + (n – 1)d] \\ &= 2a_n \end{aligned}$$

Therefore:
$2a_n = a_{n-1} + a_{n+1}$

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II. $2\ (a_3 + a_4) = (a_1 + a_2) + (a_5 + a_6)$

So we get:
$2\ (a_n + a_{n+1}) = (a_{n-2} + a_{n-1}) + (a_{n+2} + a_{n+3})$
Proof:

$2[a_1 + (n - 1)d + a_1 + nd] = a_1 + (n - 3)d + a_1 + (n - 2)d] + [a_1 + (n + 1)d + a_1 + (n + 2)d]$
$2\ (a_n + a_{n+1}) \qquad = (a_{n-2} + a_{n-1}) + (a_{n+2} + a_{n+3})$

Therefore:
$2\ (a_n + a_{n+1}) = (a_{n-2} + a_{n-1}) + (a_{n+2} + a_{n+3})$

III. $2\ (a_5 + a_6 + a_7) = (a_2 + a_3 + a_4) + (a_8 + a_9 + a_{10})$

So we get:
$2\ (a_n + a_{n+1} + a_{n+2}) = (a_{n-3} + a_{n-2} + a_{n-1}) + (a_{n+3} + a_{n+4} + a_{n+5})$

Proof:

$2[a_1 + (n - 1)d + a_1 + nd + a1 + (n + 1)d] = a_1 + (n - 4)d + a_1 + (n - 3)d + a_1 + (n - 2)d] + [a_1 + (n + 2)d + a_1 + (n + 3)d + a_1 + (n + 4)d]$

$6a_1 + 6n = 6a_1 + 6n$

Therefore:
$2\ (a_n + a_{n+1} + a_{n+2}) = (a_{n-3} + a_{n-2} + a_{n-1}) + (a_{n+3} + a_{n+4} + a_{n+5})$

General Conclusion:
$2a_m = a_p + a_q$
if $m = p + q$

Proof:
$2[a_1 + (m - 1)d] = a_1 + (p - 1)d + a_1 + (q - 1)d$
$= 2\ a_1 + (p + q - 2)d$
$= 2\ a_1 + (2m - 2)d$ $\qquad 2m = p + q$

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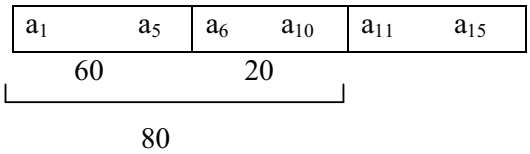
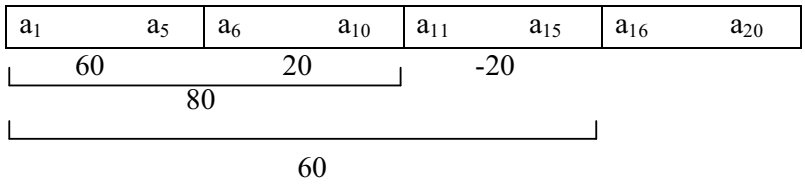
5. $S_5 = 60$ and $S_{10} = 80$. Using the conclusions from question 4, find S_{15} and S_{20}.

Solution:



From Question 4, we know that:
$2a_m = a_p + a_q$
if $m = p + q$

So:
$2(S_{10} - S_5) = S_5 + (S_{15} - S_{10})$
$2(80 - 60) = 60 + S_{15} - 80$
$2(20) = -20 + S_{15}$
$S_{15} = 60$



$2(S_{15} - S_{10}) = (S_{10} - S_5) + (S_{20} - S_{15})$
$2(60 - 80) = (80 - 60) + (S_{20} - 60)$
$-40 = S_{20} - 40$
$S_{20} = 0$

Therefore:
$S_{15} = 60$
$S_{20} = 0$

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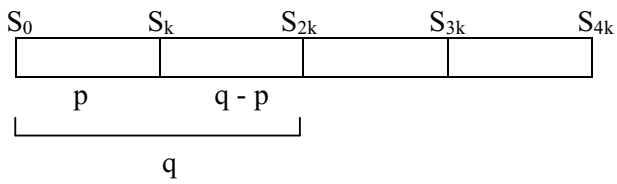
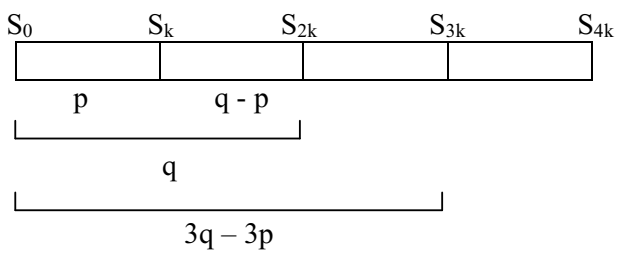
6. For an arithmetic sequence, $S_k = p$ and $S_{2k} = q$. Using the conclusions from Question 4, find S_{3k} and S_{4k}.

Solution:



From Question 4, we know that:
$2a_m = a_p + a_q$
if $m = p + q$

So:
$2(S_{2k} – S_k) = S_k + (S_{3k} – S_{2k})$
$2(q – p) = p + (S_{3k} – q)$
$2q – 2p = p + S_{3k} – q$
$S_{3k} = 3q – 3p$



$2(S_{3k} – S_{2k}) = (S_{2k} – S_k) + (S_{4k} – S_{3k})$
$2[(3q-3p)-q] = (q – p) + [S_{4k} – (3q – 3p)]$
$4q – 6p = q – p + S_{4k} – 3q + 3p$
$S_{4k} = 6q – 8p$

Therefore:
$S_{3k} = 3q – 3p$
$S_{4k} = 6q – 8p$

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7. Using the sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 find the relationship between each pair of results.

I. (3 + 11) and (5 + 9)
II. (7 + 17) and (11 + 13)
III. (5 + 19) and (9 + 15)

Then for the general arithmetic sequence $a_1, a_2, a_3, \ldots a_{n-1}, a_n$, state the general conclusion for I. II. and III.

Solution:

Let $a_1 = 3, a_2 = 5, a_3 = 7 \ldots a_{10} = 21$

I. $a_1 + a_5 = a_2 + a_4$
II. $a_3 + a_8 = a_5 + a_6$
III. $a_2 + a_9 = a_4 + a_7$

So we get:
$a_m + a_n = a_p + a_q$
if $m + n = p + q$

Proof:

From the general arithmetic sequence formula:
$a_n = a_1 + (n - 1)d$

We know:
$a_m = a_1 + (m - 1)d$
$a_p = a_1 + (p - 1)d$
$a_q = a_1 + (q - 1)d$

$$\begin{aligned} a_m + a_n &= a_p + a_q \\ &= a_1 + (p - 1)d + a_1 + (q - 1)d \\ &= 2a_1 + (p + q - 2)d \\ &= 2a_1 + (m + n - 2)d \qquad m + n = p + q \\ &= a_1 + (m - 1)d + a_1 + (n - 1)d \\ &= a_m + a_n \end{aligned}$$

Therefore:
$a_m + a_n = a_p + a_q$
if $m + n = p + q$

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8. For an arithmetic sequence, $a_5 = 8$ and $a_{36} = 32$. Using the conclusion from Question 7, find $a_{12} + a_{29}$ and $a_2 + a_{11} + a_{30} + a_{39}$.

Solution:

From Question 7, we get:
$a_m + a_n = a_p + a_q$
if $m + n = p + q$

So:
$a_5 + a_{36} = 8 + 32 = a_{12} + a_{29} =$
Because:
$5 + 36 = 40 = 12 + 29$

So:
$a_{12} + a_{29} = 40$

So:
$a_{39} + a_2 = a_{30} + a_{11} = a_{12} + a_{29} = 40$
Because:
$39 + 2 = 30 + 11 = 12 + 29 = 40$

$$\begin{aligned} a_2 + a_{11} + a_{30} + a_{39} &= a_{39} + a_2 + a_{30} + a_{11} \\ &= 2\,(a_{39} + a_2) \\ &= 2\,(a_{30} + a_{11}) \\ &= 2\,(a_{12} + a_{29}) \\ &= 2\,(40) \\ &= 80 \end{aligned}$$

$a_{39} + a_2 = a_{30} + a_{11}$

Therefore:
$a_{12} + a_{29} = 40$
$a_2 + a_{11} + a_{30} + a_{39} = 80$

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9. $a_3 + a_5 + a_7 + a_9 = 16$. Using the conclusion from Question 7, find a_6.

Solution:

From Question 7, we get:
$a_m + a_n = a_p + a_q$
if $m + n = p + q$

So:
$a_9 + a_3 = a_5 + a_7$
Because:
$9 + 3 = 5 + 7$

$a_3 + a_5 + a_7 + a_9 = 16$
$2\ (a_9 + a_3) = 16$
$2\ (a_5 + a_7) = 16$
$(a_5 + a_7) = 8$

$a_9 + a_3 = a_5 + a_7$

From Question 4, we know:
$2a_m = a_p + a_q$
if $m = p + q$

So:
$a_5 + a_7 = 2a_6 = 8$
$a_6 = 4$

Therefore:
$a_6 = 4$

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10. $a_2 + a_5 = 19$ and $S_5 = 40$. Using the conclusion from Question 7, find a_1 and a_6.

Solution:

From Question 7, we get:
$a_m + a_n = a_p + a_q$
if $m + n = p + q$

So:
$a_2 + a_5 = a_3 + a_4 = a_1 + a_6 = 19$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$
$40 = a_1 + (a_2 + a_5) + (a_3 + a_4)$
$= a_1 + 2\ (19)$
$a_1 = 2$
$a_6 = 17$ $\qquad a_1 + a_6 = 19$

Therefore:
$a_1 = 2$
$a_6 = 17$