Shady Areas
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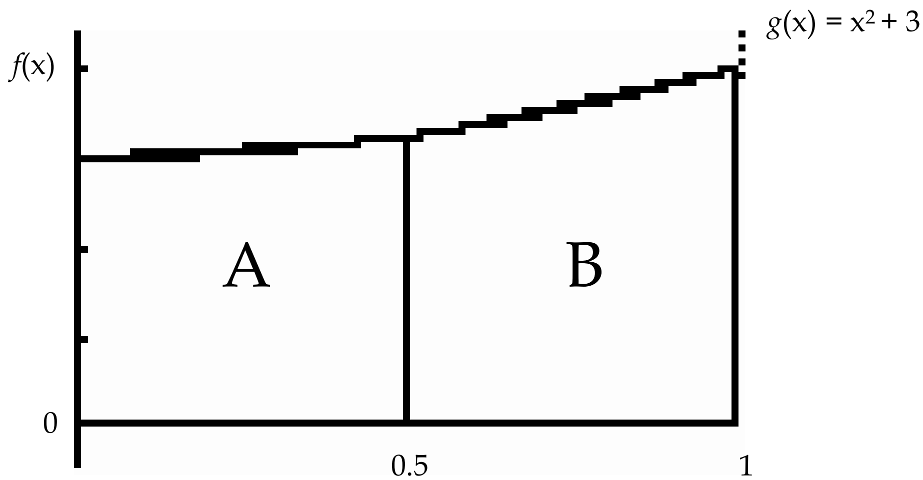
Math Methods II IB 5A
2 December 2008

In this investigation you will attempt to find a rule to approximate the area under a curve (*i.e.* between the curve and the x-axis) using trapeziums (trapezoids).

Consider the function $g(x) = x^2 + 3$

The diagram below shows the graph of *g*. The area under this curve from x = 0 to x = 1 is approximated by the sum of the area of two trapeziums.
Find this approximation.



$A = \frac{1}{2} h(b_1 + b_2)$

Trapezium A:

b_1: $g(x) = x^2 + 3$
$g(0) = (0)^2 + 3$
$= 3$

b_2: $g(x) = x^2 + 3$
$g(0.5) = (0.5)^2 + 3$
$= 3.25$

$A_A = \frac{1}{2}(0.5)(3 + 3.25)$
$A_A = 1.5625$

Trapezium B:

b_1: $g(x) = x^2 + 3$
$g(0.5) = (0.5)^2 + 3$
$= 3.25$

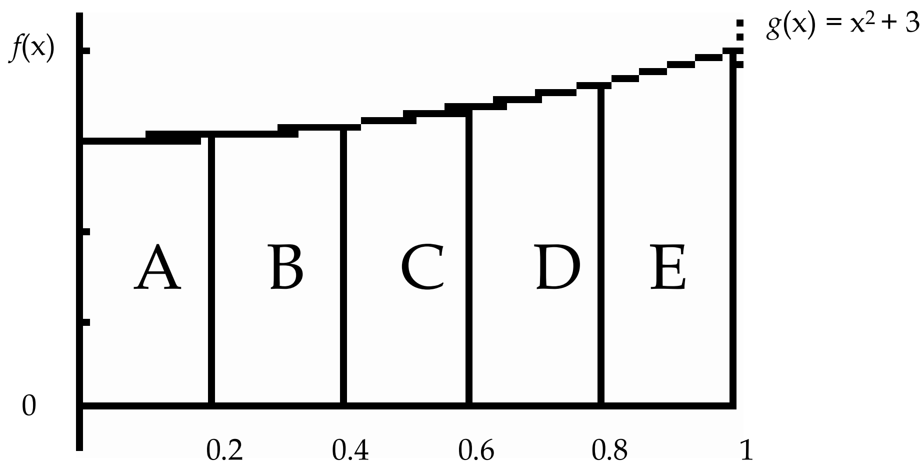
b_2: $g(x) = x^2 + 3$
$g(1) = (1)^2 + 3$
$= 4$

$A_B = \frac{1}{2}(0.5)(3.25 + 4)$

$A_B = 1.8125$

Total Area:
$A = A_A + A_B$
$A = 1.5625 + 1.8125$
$A = 3.375$

Increase the number of trapeziums to five and find a second approximation for the area.



Trapezium A:

$b_1 : g(x) = x^2 + 3$
$g(0) = (0)^2 + 3$
$= 3$

$b_2 : g(x) = x^2 + 3$
$g(0.2) = (0.2)^2 + 3$
$= 3.04$

$A_A = ½ (0.2)(3 + 3.04)$
$A_A = 0.604$

Trapezium B:

$b_1 : g(x) = x^2 + 3$
$g(0.2) = (0.2)^2 + 3$
$= 3.04$

$b_2 : g(x) = x^2 + 3$
$g(0.4) = (0.4)^2 + 3$
$= 3.16$

$A_B = ½ (0.2)(3.04 + 3.16)$
$A_B = 0.62$

Trapezium C:

$b_1 : g(x) = x^2 + 3$
$g(0.4) = (0.4)^2 + 3$
$= 3.16$

$b_2 : g(x) = x^2 + 3$
$g(0.6) = (0.6)^2 + 3$
$= 3.36$

$A_C = ½ (0.2)(3.16 + 3.36)$
$A_C = 0.652$

Trapezium D:

$b_1 : g(x) = x^2 + 3$ $\quad$ $b_2 : g(x) = x^2 + 3$

$g(0.6) = (0.6)^2 + 3$ $\quad$ $g(0.8) = (0.8)^2 + 3$

$= 3.36$ $\quad$ $= 3.64$

$$A_D = \tfrac{1}{2}(0.2)(3.36 + 3.64)$$
$$A_D = 0.7$$

Trapezium E:

$b_1 : g(x) = x^2 + 3$ $\quad$ $b_2 : g(x) = x^2 + 3$

$g(0.8) = (0.8)^2 + 3$ $\quad$ $g(1) = (1)^2 + 3$

$= 3.64$ $\quad$ $= 4$

$$A_E = \tfrac{1}{2}(0.2)(3.64 + 4)$$
$$A_E = 0.764$$

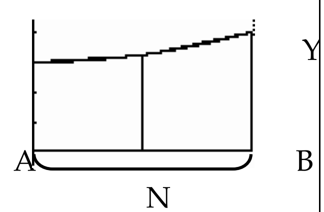
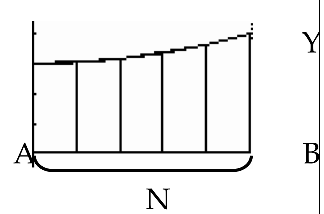
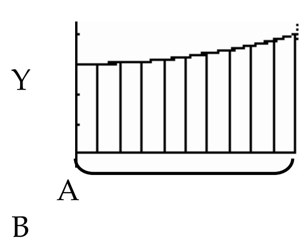
Total Area:

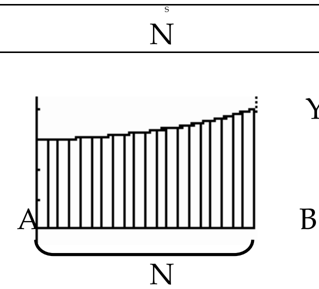
$A = A_A + A_B + A_C + A_D + A_E$

$A = 0.604 + 0.62 + 0.652 + 0.7 + 0.764$

$A = 3.34$

With the help of technology, create diagrams showing and increasing number of trapeziums. For each diagram, find the approximation for the area. What do you notice?

Number of Trapeziums	Area – Manua Calculations	Riemann Sum Application		
		Graph		SUM
2	3.375	$Y = x^2 + 3$ A = 0 B = 1 N = 2 T = 2		3.375
5	3.34	$Y = x^2 + 3$ A = 0 B = 1 N = T = 2		3.34
10		$Y = x^2 + 3$ A = 0 B = 1 N = T = 2		3.335

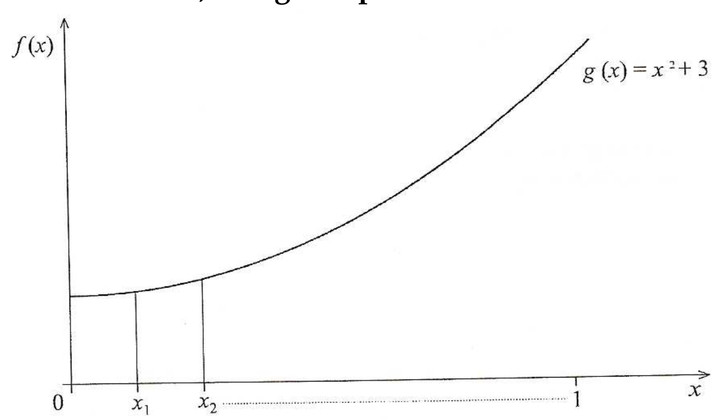
20		$Y = x^2 + 3$ $A = 0$ $B = 1$ $N = 2$ $T = 2$		3.33375

- First I calculated the area manually by hand. Then I tested these results with the use of technology, which in this case was the Riemann Sum application on a TI-84+ silver edition . Which I used to create the table above.

Observations:

The value of the sum continues to get more and more precise as the number of trapeziums under the curve increases.

Use the diagram below to find a general expression for the area under the curve of *g*, from x = 0 to x = 1, using *n* trapeziums.



$\sum = ½\, h\, [g(x_0) + g(x_1)] + ½\, h\, [g(x_1) + g(x_2)] + ½\, h\, [g(x_2) + g(x_3)] + ½\, h\, [g(x_3) + (g(x_n)]$

Two Trapeziums:

$g(x) = x^2 + 3$

from x = 0 to x = 1

$\sum = ½\, h\, [g(x_0) + g(x_1)] + ½\, h[g(x_1) + g(x_n)]$

$= ½\, (0.5)\, (3 + 3.25) + ½\, (0.5)\, (3.25 + 4)$

$= ½\, (0.5)\, [3 + 4 + (2 * 3.25)]$

$= ½ (0.5) [3 + 4 + 2(3.25)]$

Use your results to develop the general statement that will estimate the area under any curve y = *f*(x) from x = a to x = b using *n* trapeziums. Show clearly how you developed your statement.

$\sum = ½ h [g(x_0) + g(x_1)] + ½ h [g(x_1) + g(x_2)] + ½ h [g(x_2) + g(x_3)] + ½ h [g(x_3) + (g(x_n)]$

$\mathbf{h} = ½ \frac{b - a}{n} = \frac{b - a}{2n} = \frac{x_n - x_0}{2n}$

$\sum = \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2(g(x_1) + g(x_2) + g(x_3))$

$= \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

Use your general statement, with eight trapeziums, to find approximations for these areas.

$$y_1 = \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

$\sum = \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

$= \frac{3 - 1}{2(8)} [0.63 + 1.31 + 2 (0.73 + 0.83 + 0.91 + 1 + 1.08 + 1.16 + 1.24)]$

$=1.98$

$$y_2 = \frac{9x}{\sqrt{x^3 + 9}}$$

$\sum = \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

$= \frac{3 - 1}{2(8)} [2.85 + 4.5 + 2 (3.40 + 3.84 + 4.16 + 4.37 + 4.48 + 4.53 + 4.54)]$

$= 8.24625$

$$y_3 = 4x^3 - 23x^2 + 40x - 18$$

$\Sigma = \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2[\Sigma g(x_i)]$

$= \frac{3 - 1}{2(8)} [3 + 3 + 2(3.875 + 3.75 + 3 + 2 + 1.125 + 0.75 + 1.25)]$

$= 4.6875$

Find $\int_1^3 \left(\frac{x}{2}\right)^{\frac{2}{3}} dx$, $\int_1^3 \left(\frac{9x}{\sqrt{x^3+9}}\right) dx$, $\int_1^3 (4x^3 - 23x^2 + 40x - 18)\, dx$, **and compare these answers with your approximations. Comment on the accuracy of your approximations.**

$$y_1 = \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

Calculator:

$\int f(x)dx = 1.9806909$

$$y_2 = \frac{9x}{\sqrt{x^3+9}}$$

Calculator:

$\int f(x)dx = 8.2597312$

$$y_3 = 4x^3 - 23x^2 + 40x - 18$$

Calculator:

$\int f(x)dx = 4.6666667$

- My approximations are accurate because the answers that I originally got are very close to the calculator answers.

Use other functions to explore the scope and limitations of your general statement. Does it always work? Discuss how the shape of a graph influences your approximation.

- 4 Trapeziums – x = 0 to x = 1

$y = x^4 + 16x + 2$

$$\Sigma = \frac{x_n - x_0}{2n} \left(g(x_0) + g(x_n) + 2\left[\Sigma\, g(x_i)\right]\right)$$

$x_0: g(x) = x^4 + 16x + 2$

$g(0) = (0)^4 + 16(0) + 2$

$= 2$

$x_n: g(x) = x^4 + 16x + 2$

$g(1) = (1)^4 + 16(1)$

$= 19$

$x_1: g(x) = x^4 + 16x + 2$

$g(0.25) = (0.25)^4 + 16(0.25) + 2$

$= 6.00390625$

$x_2: g(x) = x^4 + 16x + 2$

$g(0.5) = (0.5)^4 + 16(0.5) + 2$

$= 10.0625$

$x_3: g(x) = x^4 + 16x + 2$

$g(0.75) = (0.75)^4 + 16(0.75) + 2$

$= 14.31640625$

$$\Sigma = \frac{1 - 0}{2(4)} \left(0 + 17 + 2\left[6.00390625 + 10.0625 + 14.31640625\right]\right)$$

$= 9.861328125$

Calculator:

$\int f(x)dx = 10.2$

$y = x^5 + 30$

$$\Sigma = \frac{x_n - x_0}{2n} \left(g(x_0) + g(x_n) + 2\left[\Sigma\, g(x_i)\right]\right)$$

$x_0: g(x) = x^5 + 30$

$g(0) = (0)^5 + 30$
$= 30$

$x_n : g(x) = x^5 + 30$
$g(1) = (1)^5 + 30$
$= 31$

$x_1: g(x) = x^5 + 30$
$g(0.25) = (0.25)^5 + 30$
$= 30.000976563$

$x_2: g(x) = x^5 + 30$
$g(0.5) = (0.5)^5 + 30$
$= 30.03125$

$x_3: g(x) = x^5 + 30$
$g(0.75) = (0.75)^5 + 30$
$= 30.23730469$

$$\Sigma = \frac{1 - 0}{2(4)} (30 + 31 + 2[30.000976563 + 30.03125 + 30.23730469]$$
$= 30.19238281$

Calculator:

[Y=] enter y = x^5 + 25 [2nd] [TRACE] [7] [ENTER] graph displayed [0] [ENTER]

[1] [ENTER]

$\int f(x)dx = 30.166667$

$$y = \frac{x + 2}{4}$$

$$\Sigma = \frac{x_n - x_0}{2n} (g(x_0) + g(x_n) + 2[\Sigma g(x_i)]$$

$x_0 : g(x) = \frac{x + 2}{4}$

$g(0) = \frac{(0) + 2}{4}$
$= ½$

$x_n : g(x) = \frac{x + 2}{4}$

$g(1) = \frac{(1) + 2}{4}$
$= 3/4$

x_1: $g(x) = \frac{x + 2}{4}$

$g(0.25) = \frac{(0.25) + 2}{4}$

$= 0.5625$

x_2 : $g(x) = \frac{x + 2}{4}$

$g(0.5) = \frac{(0.5) + 2}{4}$

$= 0.625$

x_3 : $g(x) = \frac{x + 2}{4}$

$g(0.75) = \frac{(0.75) + 2}{4}$

$= 0.6875$

$\sum = \frac{1 - 0}{2(4)} (1/2 + 3/4 + 2[0.5625 + 0.625 + 0.6875]$

$= 0.625$

Calculator:

$\int f(x)dx = 0.625$

Conclusion:

My original calculations from my general statement seemed to be very close to the answers that I later found using a calculator. I do not believe that the shapes of the graphs have an effect on the approximation