# **Creating a logistic model**
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## Theory

A logistic model is expressed as:

$$u_{n+1} = ru \qquad \{1\}$$

The growth factor *r* varies according to *u* . If r=1 then the population is stable.

## Solution

1. *A hydroelectric project is expected to create a large lake into which some fish are to be placed. A biologist estimates that if $1x10^4$ fish were introduced into the lake, the population of fish would increase by 50% in the first year, but the long-term sustainability limit would be about $6\ x10^4$. From the information above, write two ordered pairs in the form $(u_0, r_0)$, $(u_0, r_0)$ where $U_n=6x10^4$. Hence, determine the slope and equation of the linear growth in terms of $U_n$*

As $U_n=6$ 10 , the population in the lake is stable. Thus from the definition of a logistic model, *r* must equal to 1 as $u_n$ approaches the limit. If the growth in population of fish (initially 1 10 ) is 50% during
the first year, *r* must be equal to 1.5. Hence the ordered pairs are:

(1 , .5) , ,
)

One can graph the two ordered pairs. It has been requested to find the growth factor *in terms of $U_n$*, and thus one should graph the population (x-axis) *versus* the growth factor (y-axis):

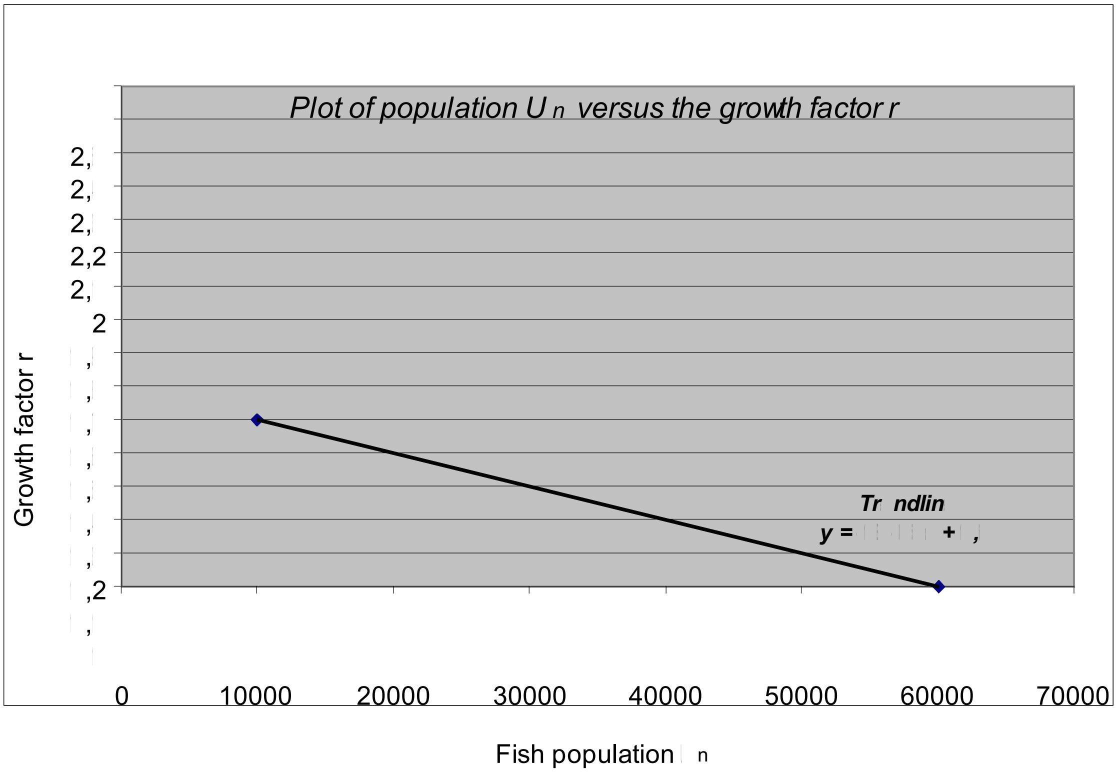

**Figure 1.1.** Graphical plot of the fish population $U_n$ versus the growth factor *r* of the logistic model $u_{n+1} =$

The slope of the trend line in figure 1 is determined graphically. The equation of the trend line is of the form *y=mx + b*: $\underline{y = -1 \quad 10^{-5} x + 1.6}$. Thus m$= -1 \quad 10^{-5}$. This can be verified algebraically:

$$m = \frac{\Delta y}{x} = \frac{1.0 \quad 1.5}{} = \quad 1 \quad 10^{\ 5}$$

The linear growth factor is said to depend upon $u_n$ and thus a linear equation can be written:

$$r_n = mu_n + c$$
$$= (-1 \quad 10^{-})(u_n) + c$$
$$1.5 = (-1 \quad 10^{-})(1 \quad 10\ ) + c$$
$$® \ c = 1.5 - (-1 \quad 10^{-5})(1 \quad 10^{4}) = 1.6$$

Hence the equation of the linear growth factor is:

$$\underline{r_n = \quad 0^{\ 5}\ _n + 1.6} \qquad \{2\}$$

2. *Find the logistic function model for $u_{n+1}$*

Using equations {1} and {2}, one can find the equation for $u_{n+1}$:

$$u_{n+1} = ru$$
$$r_n = -1 \quad 10^{-5} u_n + 1.6$$
$$\therefore u \quad = (- \quad 0^{-5}\ _n + 1.6) u_n$$
$$u$$
$$= (-1 \quad 10^{-})(u_n)(u_n) + 1.6 u_n$$
$$= (-1 \quad 10^{-})(u_n^{\ 2}) + 1.6 u_n$$

The logistic function model for $u_{n+1}$ is:

$$\underline{u_{n+1} = \quad 1 \quad 10^{-5} \quad ^2) + 1.6 u_n} \qquad \{3\}$$

3. *Using the model, determine the fish population over the next 20 years and show these values using a line graph*

One can determine the population of the first 20 years just by knowing that and

$u = 1 \quad 10$

$u = 1.5 \quad 10$. For instance $u_3$ is

$$u_3 = (-1 \quad 10^{-5})(u_2^{\ 2}) + 1.6 u_2$$
$$= (-1 \quad 10^{-})(1. \quad 10^{4})^2 + 1.6(1. \quad 10^{4})$$
$$= 2.18$$
$$10^{4}$$

**Table 3.1.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {3}. The interval of calculation is 1 year.

| Year | Population | Year | Population |
|---|---|---|---|
| 1 | 1.0 10 | 11 | 5.98 10 |
| 2 | 1.5 10 | 12 | 5.99 10 |
| 3 | 2.18 10 | 13 | 5.996 10 |
|  | 3.01 10 | 1 | 5.999 10 |
| 5 | 3.91 10 | 15 | 5.999 10 |
| 6 | .72 10 | 16 | 5.9997 10 |

| | | | |
|---|---|---|---|
| 7 | $5.34\ \ 10^{4}$ | 17 | $5.9999\ \ 10^{4}$ |

| | | | |
|---|---|---|---|
| 8 | 5.69 10 | 18 | 5.9999 10 |
| 9 | 5.86 10 | 19 | 6.0 10 |
| 10 | 5.9 10 | 20 | 6.0 10 |

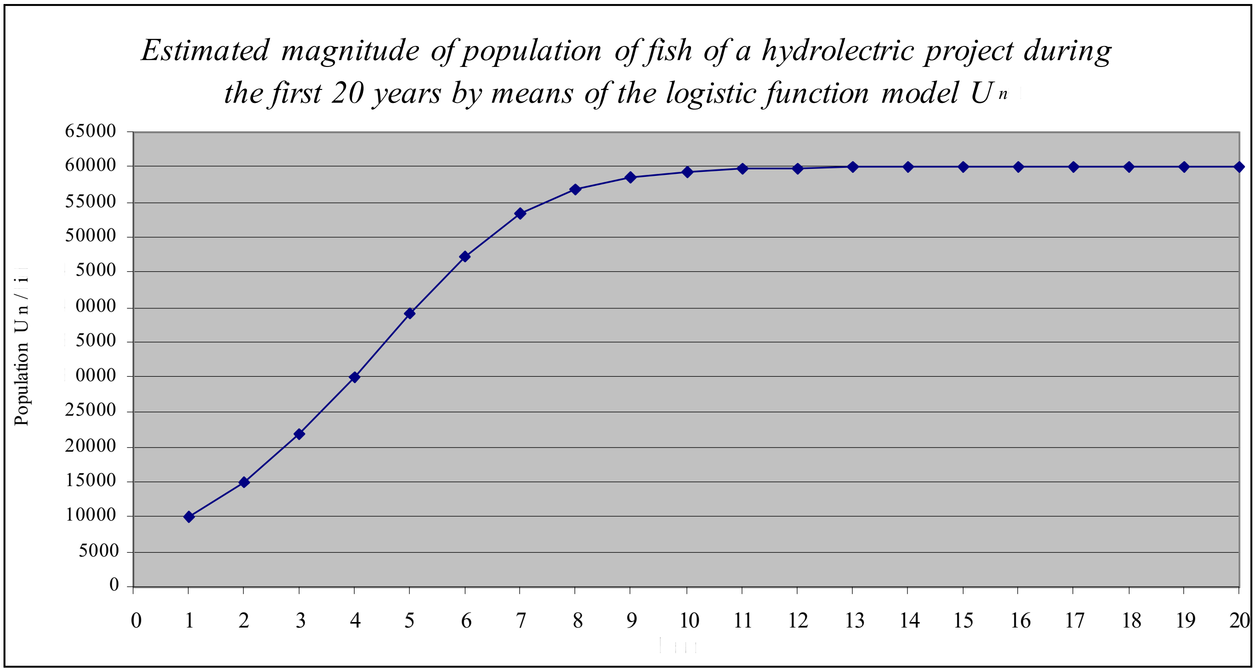

**Figure 3.1.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model {3}. The graph is asymptotic approaching 6.0x$10^4$ fish despite the years 19 and 20 showing data rounded off to 6.0x$10^4$.

4. *The biologist speculates that the initial growth rate may vary considerably. Following the process above, find new logistic function models for $u_{n+1}$ using the initial growth rates r=2, 2.3 and 2.5. Describe any new developments.*

   ***a.*** For *r=2*. One can write two ordered pairs , 2. ) , , ). The graph of the two ordered pairs is:

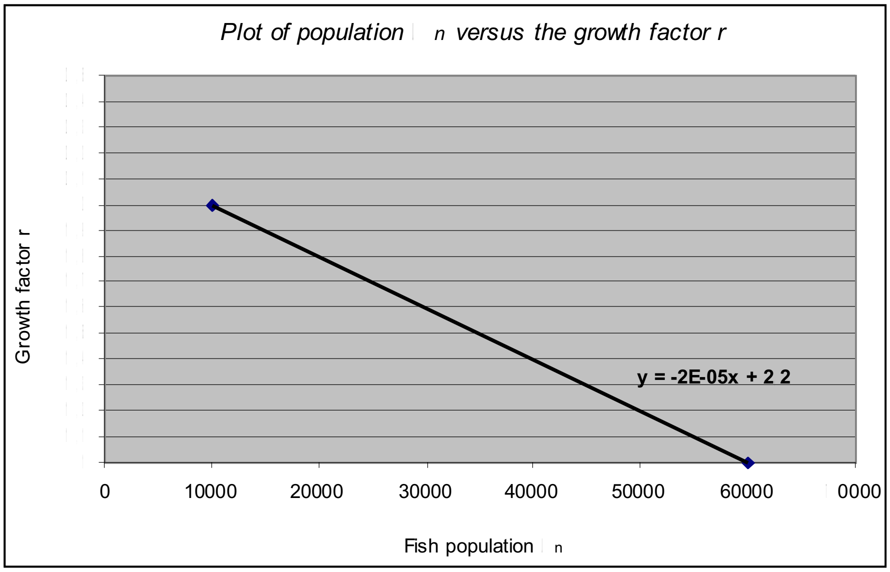

**Figure 4.1.** Graphical plot of the fish population $U_n$ versus the growth factor *r* of the logistic model $u_{n+1} =$

The equation of the trend line is of the form *y*=*mx* + *b*: $y = -2 \quad 10^{-5} x + 2.2$. Thus $\quad -2 \quad 10^{-5}$. This can be verified algebraically:

m=

$$m = \frac{\Delta y}{\Delta x} = \frac{1.0 -}{\quad 1 \ ^{4} - 1} = - \quad 1 \ ^{-5}$$

$$1 \ ^{4}$$

The linear growth factor is said to depend upon $u_n$ and thus a linear equation can be written:

$$r_n = mu_n + c$$

$$= (-2 \quad 10^{-})(u_n) + c$$

$$2.0 = (-2 \quad 10^{-})(1 \quad 10 \ ) + c$$

$$® \ c = 2.0 - (-2 \quad 10^{-5})(1 \quad 10^{4}) = 2.2$$

Hence the equation of the linear growth factor is:

$$r_n = \quad 10^{5} \ _n + 2.2 \qquad \{4\}$$

Using equations {1} and {2}, one can find the equation for $u_{n+1}$:

$$u_{n+1} = ru$$

$$r_n = -2 \quad 10^{-5} u_n + 2.2$$

$$\therefore u \quad = (-2 \quad 0^{-} \ _n + 2.2)u_n$$

$$u$$

$$= (-2 \quad 10^{-})(u_n)(u_n) + 2.2u_n$$

$$= (-2 \quad 10^{-})(u_n^{\ 2}) + 2.2u_n$$

The logistic function model for $u_{n+1}$ is:

$$\underline{u_{n+1} = \quad 1 \ ^{-5} \quad ^{2}) + 2.2u_n} \qquad \{5\}$$

***b.*** For *r*=*2.3*. One can write two ordered pairs $\quad , 2.3)$, $\quad , \ )$. The graph of the two ordered pairs is:

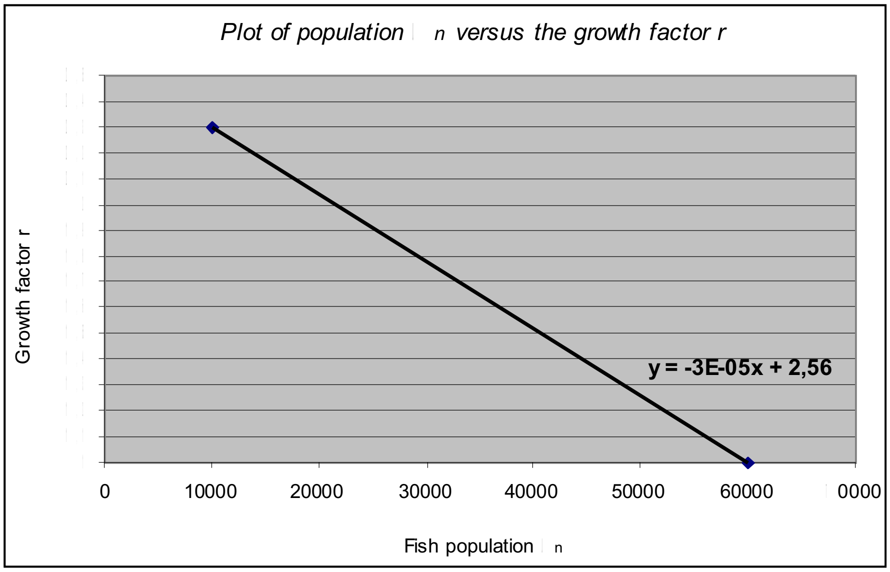

**Figure 4.2.** Graphical plot of the fish population $U_n$ versus the growth factor $r$ of the logistic model $u_{n+1} =$

The equation of the trend line is of the form $y=mx + b$: $y = -3 \quad 10^{-} x + 2.56$. Thus $m= -2 \quad 10^{-5}$. This can be verified algebraically:

$$m = \frac{\Delta y}{\Delta x} = \frac{1.0 -}{10^4 - 1 \atop 10^4} = - \quad 10^{-}$$

The difference surges because the program used to graph rounds $m$ up to one significant figure.

The linear growth factor is said to depend upon $u_n$ and thus a linear equation can be written:

$$r_n = mu_n + c$$
$$= (-2.6 \quad 10^{-5})(u_n) + c$$
$$2.3 = (-2.6 \quad 10^{-})(1 \quad 10\ ) + c$$
$$® \ c = 2.3 - (-2.6 \quad 10^{-})(1 \quad 10^4) = 2.\ 6$$

Hence the equation of the linear growth factor is:

$$r_n = \quad {}_n + 2.56 \qquad \{6\}$$

Using equations {1} and {2}, one can find the equation for $u_{n+1}$:

$$u_{n+1} = ru$$
$$r_n = -2.6 \quad 10^{-} u_n + 2.56$$
$$\therefore u_{n+1} = (-2.6 \quad 10^{-} \quad {}_n + 2.56)u_n$$
$$u$$
$$= (-2.6 \quad 10^{-5})(u_n)(u_n) + 2.56u_n$$
$$= (-2.6 \quad 10^{-5})(u_n^{\ 2}) + 2.56u_n$$

The logistic function model for $u_{n+1}$ is:

$$\underline{u_{n+1} = \quad 1^{-5} \quad {}^2) + 2.56u_n} \qquad \{7\}$$

**c.** For *r=2.5.* One can write two ordered pairs ( , 2.5) , ( , ). The graph of the two ordered pairs is:

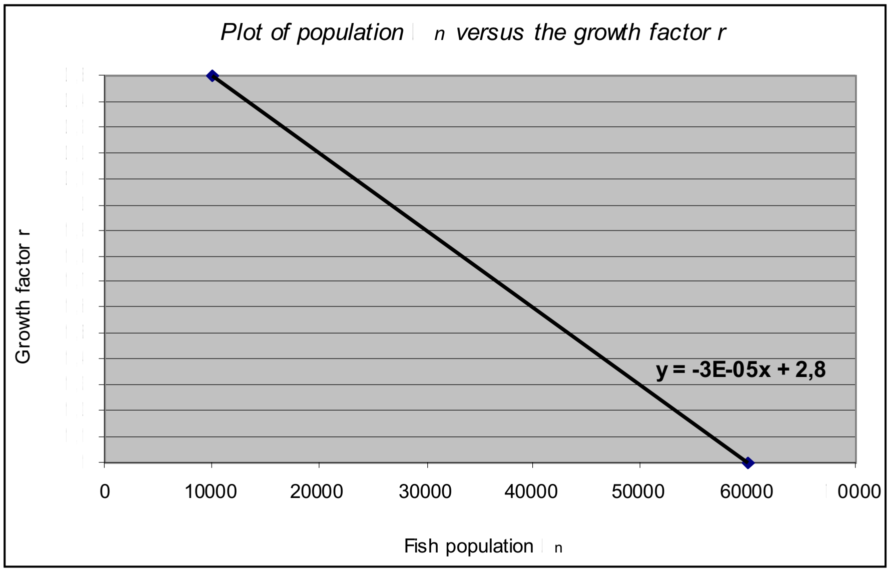

**Figure 4.3.** Graphical plot of the fish population $U_n$ versus the growth factor $r$ of the logistic model $u_{n+1} =$

The equation of the trend line is of the form $y=mx+b$: $y=-3\quad 10^{-}\,x+2.8$. Thus $m=$ $-3\quad 10^{-}$. This can be verified algebraically:

$$m=\frac{\Delta y}{\Delta x}=\frac{1.0-}{10^4-1}=-3\;0\quad 10^{-}$$

$$10^4$$

The linear growth factor is said to depend upon $u_n$ and thus a linear equation can be written:

$$r_n = mu_n + c$$
$$=(-3.0\quad 10^{-5})(u_n)+c$$
$$2.5=(-3.0\quad 10^{-}\ )(1\quad 10\ )+c$$
$$® \ c=2.5-(-3.0\quad 10^{-5})(1\quad 10^{4})=2.8$$

Hence the equation of the linear growth factor is:

$$r_n = \quad _n+2.8 \qquad \{8\}$$

Using equations {1} and {2}, one can find the equation for $u_{n+1}$:

$$u_{n+1}=ru$$
$$r_n=-3.0\quad 10^{-}\,u_n+2.8$$
$$\therefore u_{n+1}=(-3.0\quad 10^{-}\quad _n+2.8)u_n$$
$$=(-3.0\quad 10^{-5})(u_n)(u_n)+2.8u_n$$
$$=(-3.0\quad 10^{-5})(u_n{}^2)+2.8u_n$$

The logistic function model for $u_{n+1}$ is:

$$\underline{u_{n+1}=\quad 1^{-5}\quad {}^2)+2.8u_n} \qquad \{9\}$$

The findings for a, b and c enables one to calculate the magnitude of the population for three studies with different growth factor. These results are shown schematically and graphically:

**Table 4.1. Growth factor r=2.0** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {5}. The interval of calculation is 1 year. Please note that from year 6 and onwards the population resonates above and below the limit yet due to the rounding up of numbers this is not observed), and it finally stabilizes in year 17 and onwards) where the population is exactly equal to the sustainable limit.

| Year | Population | Year | Population |
|---|---|---|---|
| 1 | 1.0 10 | 11 | 6.00 10 |
| 2 | 2.0 10 | 12 | 6.00 10 |
| 3 | 3.60 10 | 13 | 6.00 10 |
|  | 5.33 10 | 1 | 6.00 10 |
| 5 | 6.0 10 | 15 | 6.00 10 |
| 6 | 5.99 10 | 16 | 6.00 10 |
| 7 | 6.00 10 | 17 | 6.00 10 |
| 8 | 6.00 10 | 18 | 6.00 10 |
| 9 | 6.00 10 | 19 | 6.00 10 |
| 10 | 6.00 10 | 20 | 6.00 10 |

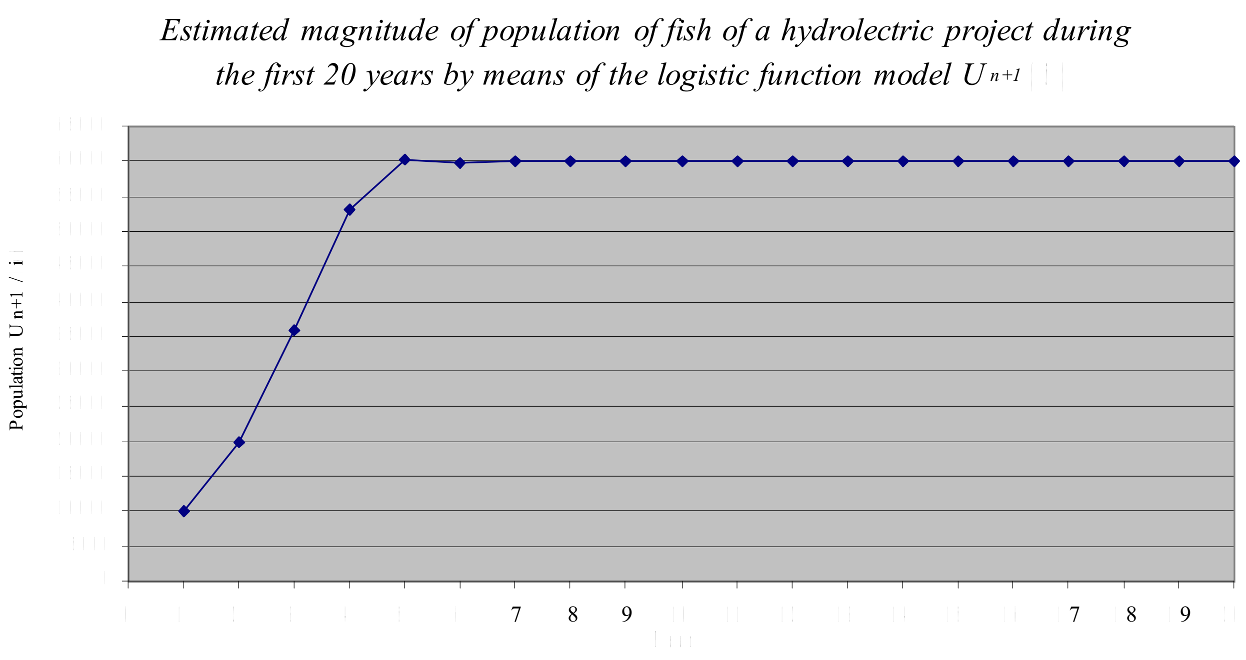

**Figure 4.1. Growth factor r=2.0.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model 5 .

**Table 4.2. Growth factor r=2.3** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {7}. The interval of calculation is 1 year.

| Year | Population | Year | Population |
|---|---|---|---|
| 1 | 1.0 10 | 11 | 6.00 10 |
| 2 | 2.3 10 | 12 | 6.00 10 |
| 3 | .51 10 | 13 | 6.00 10 |
|  | 6.26 10 | 1 | 6.00 10 |
| 5 | 5.8 10 | 15 | 6.00 10 |
| 6 | 6.08 10 | 16 | 6.00 10 |
| 7 | 5.95 10 | 17 | 6.00 10 |
| 8 | 6.03 10 | 18 | 6.00 10 |
| 9 | 5.98 10 | 19 | 6.00 10 |
| 10 | 6.01 10 | 20 | 6.00 10 |

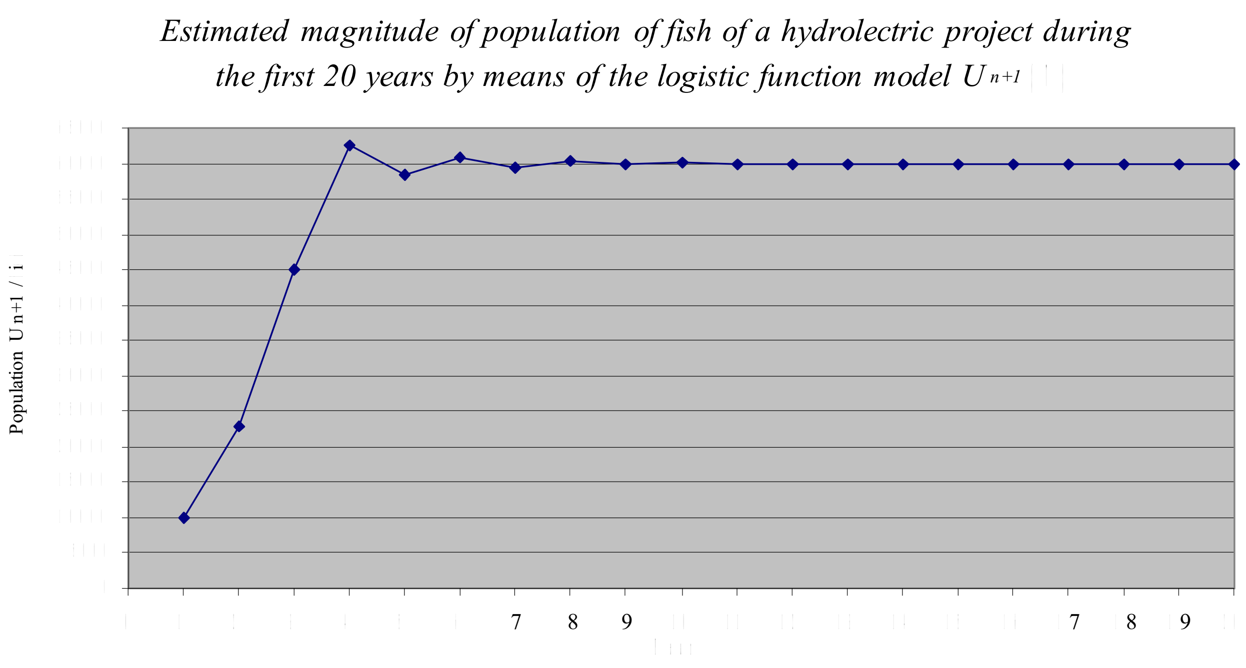

**Figure 4.2. Growth factor r=2.3.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model 7 .

**Table 4.3. Growth factor r=2.5.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model 9 . The interval of calculation is 1 year.

| Year | Population | Year | Population |
|---|---|---|---|
| 1 | 1.0 10 | 11 | 5.90 10 |
| 2 | 2.5 10 | 12 | 6.08 10 |
| 3 | 5.13 10 | 13 | 5.9 10 |
|  | 6. 7 10 | 1 | 6.05 10 |
| 5 | 5.56 10 | 15 | 5.96 10 |
| 6 | 6.30 10 | 16 | 6.03 10 |
| 7 | 5.7 10 | 17 | 5.97 10 |

| | | | |
|---|---|---|---|
| 8 | 6.19 10 | 18 | 6.02 10 |
| 9 | 5.8 10 | 19 | 5.98 10 |
| 10 | 6.12 10 | 20 | 6.01 10 |

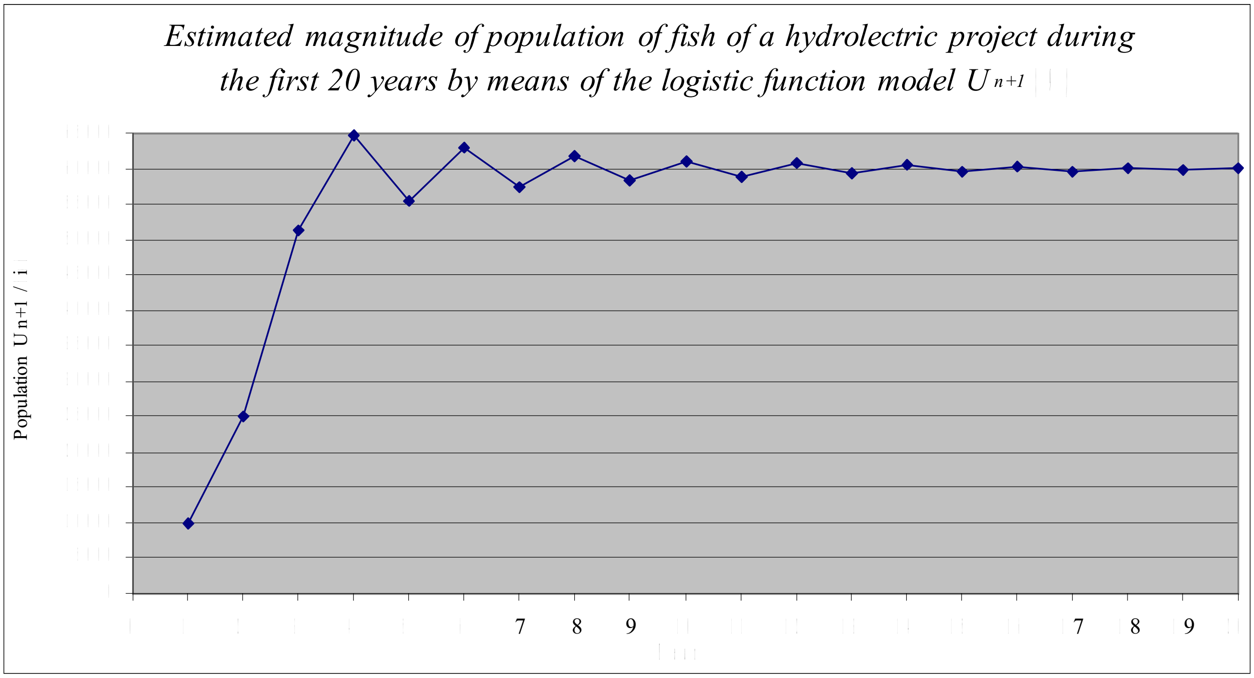

**Figure 4.3. Growth factor r=2.5.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model 9 .

The logistic function models attain greater values for *m* and *c* when the growth factor *r* increases in magnitude. As a consequence, the initial steepness of graphs 4.1, 4.2 and 4.3 increases, and thus the growth of the population increase. Also, as these values increase the population (according to this model) goes above the sustainable limit but then drops below the subsequent year so as to stabilize. There is no longer an asymptotic relationship (going from {5} to {9}) but instead a stabilizing relationship where the deviation from the sustainable limit decrease until remaining very close to the limit.

The steepness of the initial growth is largest for figure 4.3, and so is the deviation from the long-term sustainability limit of $6x10^4$ fish.

5. *A peculiar outcome is observed for higher values of the initial growth rate. Show this with an initial growth rate of r=2.9. Explain the phenomenon.*

One can start by writing two ordered pairs , 2.9) , , )

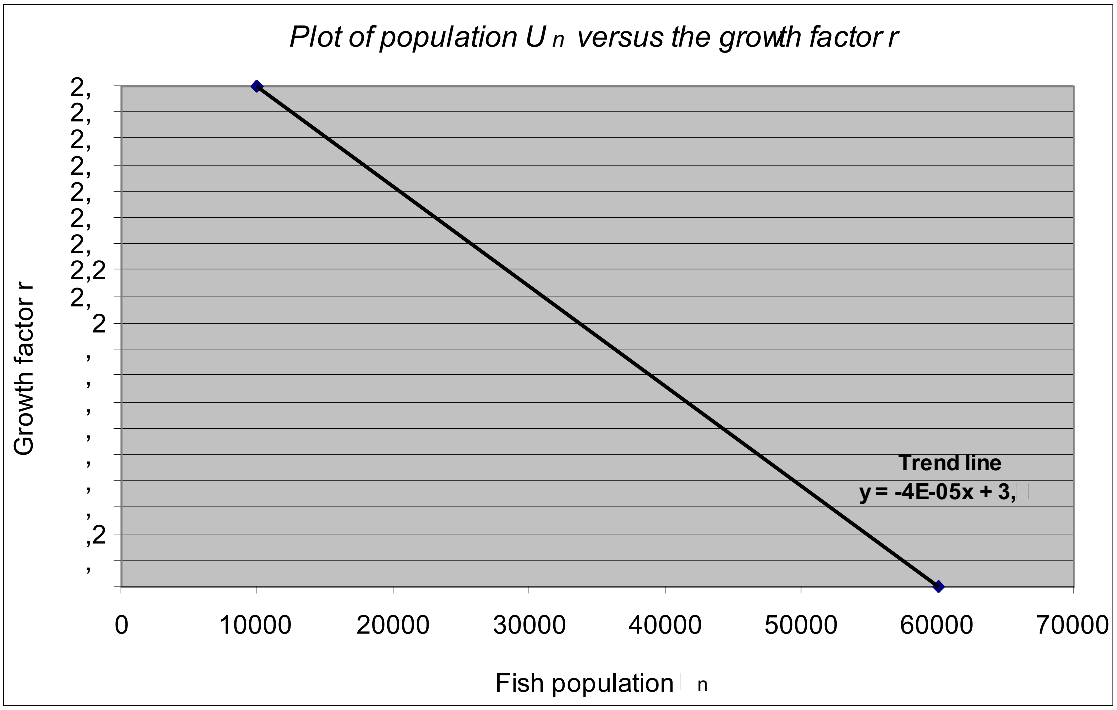

**Figure 5.1.** Graphical plot of the fish population $U_n$ versus the growth factor $r$ of the logistic model $u_{n+1} =$

The equation of the trend line is of the form $y=mx+b$: $y = -4 \quad 10^{-5}x + 3.28$. Thus $-4 \quad 10^{-5}$.
m=
This can be verified algebraically:

$$m = \frac{\Delta y}{\Delta x} = \frac{1.0 -}{10^4 - 1} = -3\ 8 \quad 10^{-}$$

$$10^4$$

Once again the use of technology is somewhat limiting because the value of $m$ is given to one significant figure. Algebraically this can be amended.

The linear growth factor is said to depend upon $u_n$ and thus a linear equation can be written:

$$\begin{aligned} r_n &= mu_n + c \\ &= (-3.8 \quad 10^{-})(u_n) + c \\ 2.9 &= (-3.8 \quad 10^{-})(1 \quad 10\ ) + c \\ ® \ c &= 2.9 - (-3.8 \quad 10^{-5})(1 \quad 10^4) = 3.28 \end{aligned}$$

Hence the equation of the linear growth factor is:

$$r_n = \quad {}_n + 3.28 \qquad \{10\}$$

Using equations {1} and {2}, one can find the equation for $u_{n+1}$:

$$u_{n+1} = ru$$

$$r_n = -3.8 \quad 10^{-} \; u_n + 3.28$$

$$\therefore u \quad = (-3.8 \quad 0^{-5}{}_n + 3.28)u_n$$

$$u$$

$$= (-3.8 \quad 10^{-})(u_n)(u_n) + 3.28u_n$$

$$= (-3.8 \quad 10^{-})(u_n{}^{2}) + 3.28u_n$$

The logistic function model for $u_{n+1}$ is:

$$\underline{u_{n+1} = \quad 1^{-5} \quad {}^{2}) + 3.28u_n} \qquad \{11\}$$

**Table 4.1. Growth factor r=2.9.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {11}. The interval of calculation is 1 year.

| Year | Population | Year | Population |
|---|---|---|---|
| 1 | 1.0 10 | 11 | 7.07 10 |
| 2 | 2.9 10 | 12 | .19 10 |
| 3 | 6.32 10 | 13 | 7.07 10 |
|  | 5.56 10 | 1 | .19 10 |
| 5 | 6. 9 10 | 15 | 7.07 10 |
| 6 | 5.28 10 | 16 | .19 10 |
| 7 | 6.73 10 | 17 | 7.07 10 |
| 8 | .87 10 | 18 | .19 10 |
| 9 | 6.96 10 | 19 | 7.07 10 |
| 10 | . 2 10 | 20 | .19 10 |

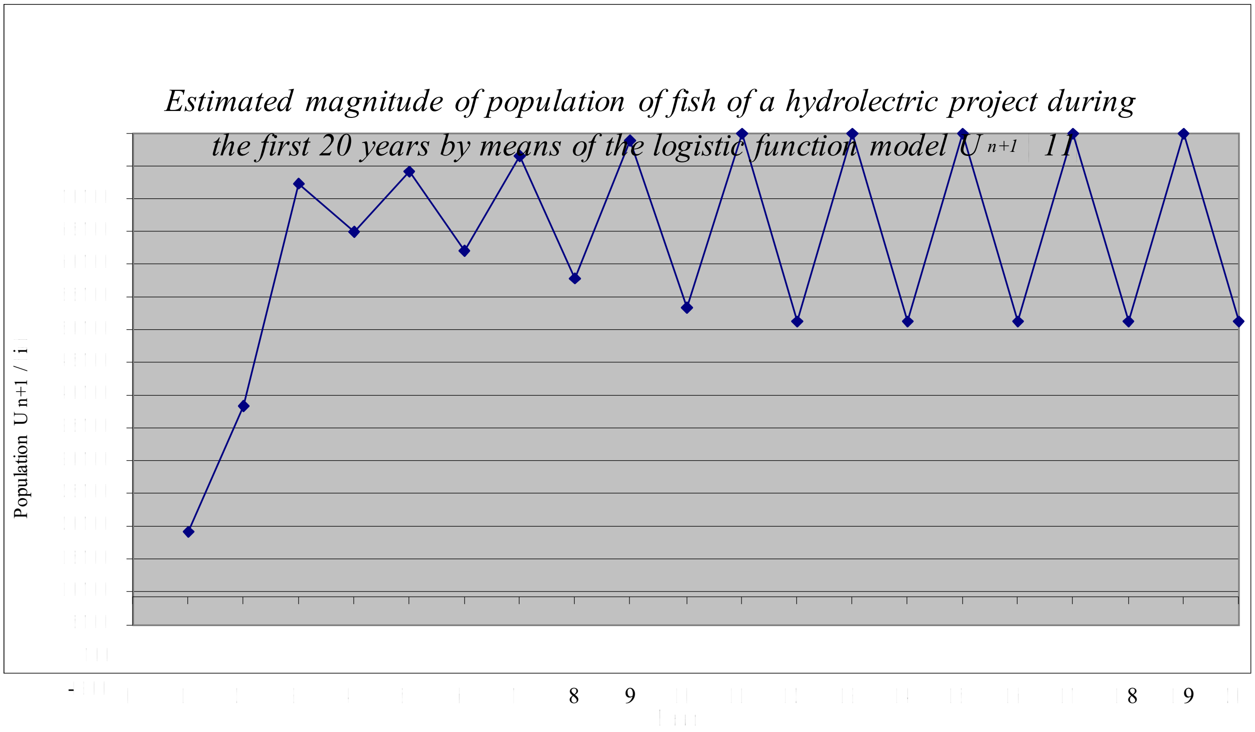

**Figure 5.2. Growth factor r=2.9.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model {11}.

With a considerably large growth rate, *r=2.9*, the long-term sustainable limit is reached after already 3 years, and since this limit cannot be surpassed over a long time, the fourth year is characterized by a drop in population (ie. the death of roughly thirty thousand fish), followed by a demographic explosion. This continues on until two limits are reached, with a maximum and a minimum value. Hence a ***dynamic system*** is established. Instead of an asymptotic behaviour it goes from the maximum to the minimum [between the very low ($4.19 \times 10^4$) and the excessively high ($7.07 \times 10^4$)].

As previously explained, the population *cannot* remain that high and must thus stabilize itself, yet this search for stability only leads to another extreme value. The behaviour of the population is opposite to that of an equilibrium in the population (rate of birth=rate of death) as the rates are first very high, but are then very low in the next year. This means that in year 11 the rate of deaths is very high (or that of birth very low, or possibly both) and hence the population in year 12 is significantly lower.

Another aspect to consider when explaining this movement of the extrema is the growth factor. Once the two limits are established *r* ranges as follows: $r \in [0.593\ ,\ 1.69]$ and it attains the value r=0.593 when the population is at its maximum going towards its minimum, and the value r=0.1.69 when the population is at its minimum going towards its maximum.

6. *Once the fish population stabilizes, the biologist and the regional managers see the commercial possibility of an annual controlled harvest. The difficulty would be to manage a sustainable harvest without depleting the stock. Using the first model encountered in this task with r=1.5, determine whether it would be feasible to initiate an annual harvest of* 5 10 *fish after a stable population is reached. What would be the new, stable fish population with an annual harvest of this size?*

From figure 3.1 it could be said that the population stabilizes at around year 14 (due to a round-off). Most correctly this occurs in year 19 where the population strictly is equal to the long-term sustainable limit. Nevertheless, if one began a harvest of 5 10 fish at any of these years (depending on what definition of 'stable' is employed), one could say that $u_{n+1}$ is reduced by 5 10 , so that when one is calculating the growth of the population at the end of year 1, one takes into account that 5 10 fish has been harvested. Mathematically this is shown as:

$$u_{n+1} = (-1 \quad 10^{-5})(u_n + 1.6(u_n)^2 - 5 \quad 10 \qquad \{12\}$$

**Table 6.1.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {12}. A harvest of fish is made per annum

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| | 6 | | 5 |
| 2 | 5 55 | 2 | 5 |
| | 5 28 | | 5 |
| | 5 6 | | 5 |

| | | | |
|---|---|---|---|
| 5 | 5.09 10 | 15 | 5.00 10 |

| | | | |
|---|---|---|---|
| 6 | 5.05 10 | 16 | 5.00 10 |
| 7 | 5.03 10 | 17 | 5.00 10 |
| 8 | 5.01 10 | 18 | 5.00 10 |
| 9 | 5.01 10 | 19 | 5.00 10 |
| 10 | 5.01 10 | 20 | 5.00 10 |

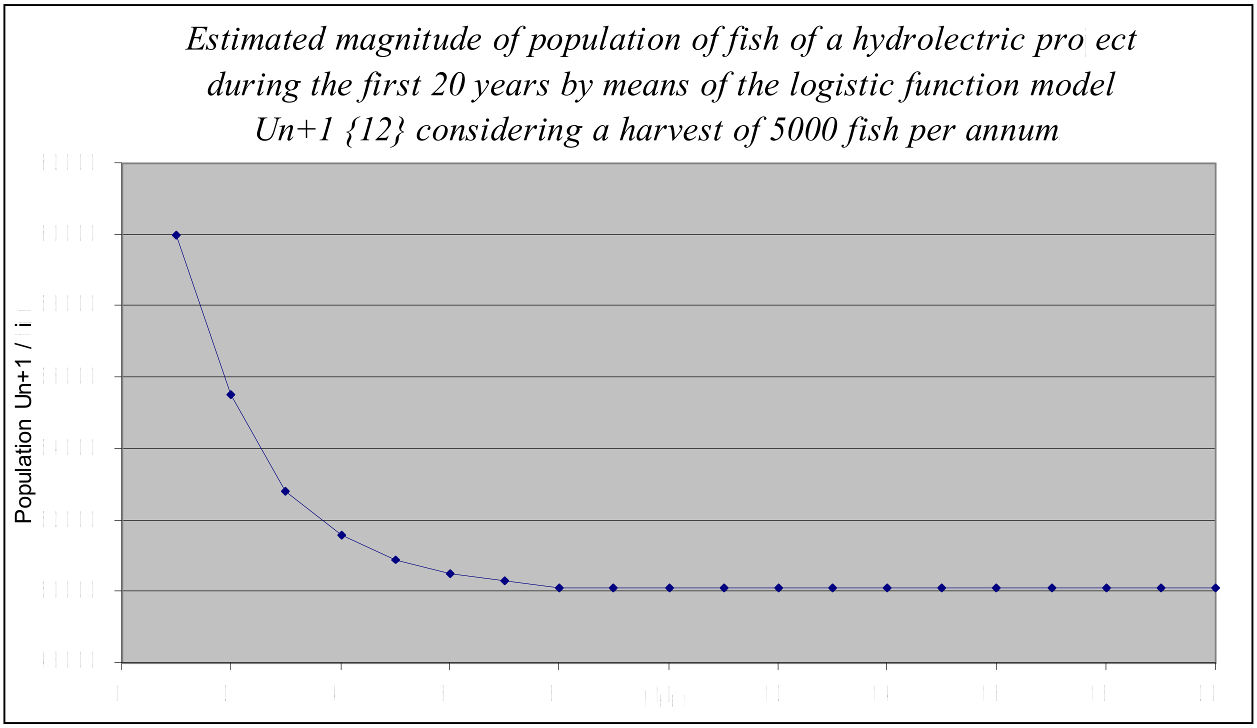

**Figure 6.1.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model {12}. The model considers an annual harvest of 5 $10^3$ fish. The stable fish population is 5.0 10 .

The behaviour observed in figure 6.1 resembles an inverse square relationship, and although there is no physical asymptote, the curve appears to approach some y-value, until it actually attains the y-coordinate 5.0 10 . **<u>It is feasible</u>** to initiate an annual harvest of 5 10 fish *after* attaining a stable population (when that occurs has already been discussed). From the graph one can conclude that the annual stable fish population with an annual harvest of 5.0 10 fish is **<u>5.0x10⁴</u>**.

7. *<u>Investigate other harvest sizes. Some annual harvests will cause the populations to die out. Illustrate your findings graphically.</u>*

   ***a.*** Consider a harvest of 7.5 10 . Then the logistic function model is:

$$u_{n+1} = (-1\ \ 10^{5})(u_n^2 + 1.6(u_n)) - 7.5\ \ 10 \qquad \{13\}$$

**Table 7.1.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {13}. A harvest of 7.5 10 fish is made per annum

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|

| | | | |
|---|---|---|---|
| 1 | 6.0 10 | 16 | .2 10 |
| 2 | 5.25 10 | 17 | .23 10 |
| 3 | .89 10 | 18 | .23 10 |
| | .69 10 | 19 | .23 10 |
| 5 | .55 10 | 20 | .23 10 |
| 6 | . 6 10 | 21 | .23 10 |
| 7 | . 0 10 | 22 | .23 10 |
| 8 | .35 10 | 23 | .23 10 |
| 9 | .32 10 | 2 | .23 10 |
| 10 | .30 10 | 25 | .23 10 |
| 11 | .28 10 | 26 | .23 10 |
| 12 | .26 10 | 27 | .23 10 |
| 13 | .25 10 | 28 | .23 10 |
| 1 | .25 10 | 29 | .23 10 |
| 15 | 5.2 10 | 30 | .22 10 |

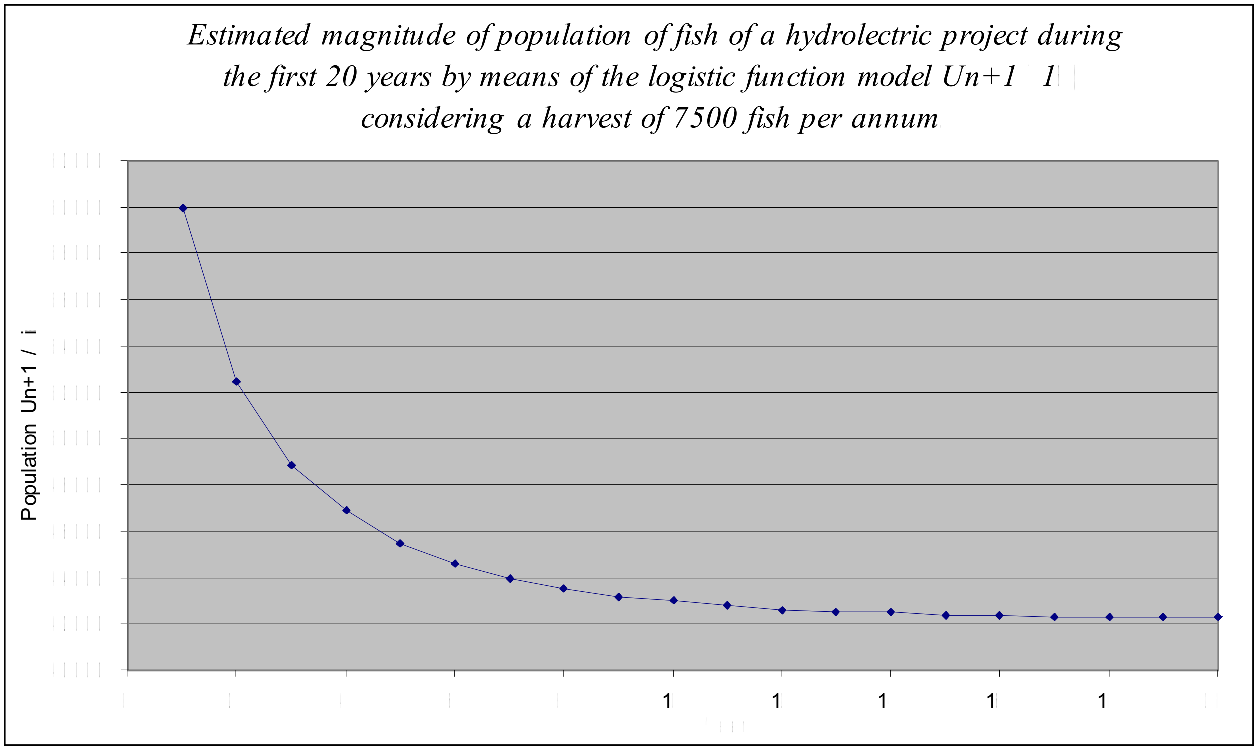

**Figure 7.1.** Graphical plot of the fish population of the hydroelectric project on an interval of 20 years using logistic function model {13}. The model considers an annual harvest of $7.5 \quad 10$ fish. The stable fish population is $4.97 \quad 10^4$.

From figure 7.1 one can observe this behaviour resembling a hyperbolic behaviour ($x^{-1}$). One can conclude that **it is feasible** to initiate an annual harvest of $7.5 \quad 10$ fish *after* attaining a stable population

(when that occurs has already been discussed). The curve approaches (and becomes) the value of **$4.22x10^4$ fish**, which is the annual stable fish population with an annual harvest of 7.5  10  fish.

***b.*** Consider a harvest of 1  $10^4$ fish. Then the logistic function model is:

$$u_{n+1} = \frac{(-1 \quad 1\hat{0} \quad )(u_n^2 + 1.6(u_n) \quad -1 \qquad {}^{4}}{) \qquad\qquad\qquad\qquad 10} \qquad \{14\}$$

**Table 7.2.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {14}. A harvest of $1x10^4$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 6.00 10 | 16 | 2.61 10 |
| 2 | 5.00 10 | 17 | 2. 9 10 |
| 3 | .50 10 | 18 | 2.37 10 |
|  | .18 10 | 19 | 2.23 10 |
| 5 | 3.9 10 | 20 | 2.07 10 |
| 6 | 3.75 10 | 21 | 1.88 10 |
| 7 | 3.59 10 | 22 | 1.65 10 |
| 8 | 3. 6 10 | 23 | 1.37 10 |
| 9 | 3.3 10 | 2 | 1.00 10 |
| 10 | 3.2 10 | 25 | 5.07 $10^3$ |
| 11 | 3.12 10 | 26 | -2.15 $10^3$ |
| 12 | 3.02 10 | 27 | - |
| 13 | 2.92 10 | 28 | - |
| 1 | 2.82 10 | 29 | - |
| 15 | 2.72 10 | 30 | - |

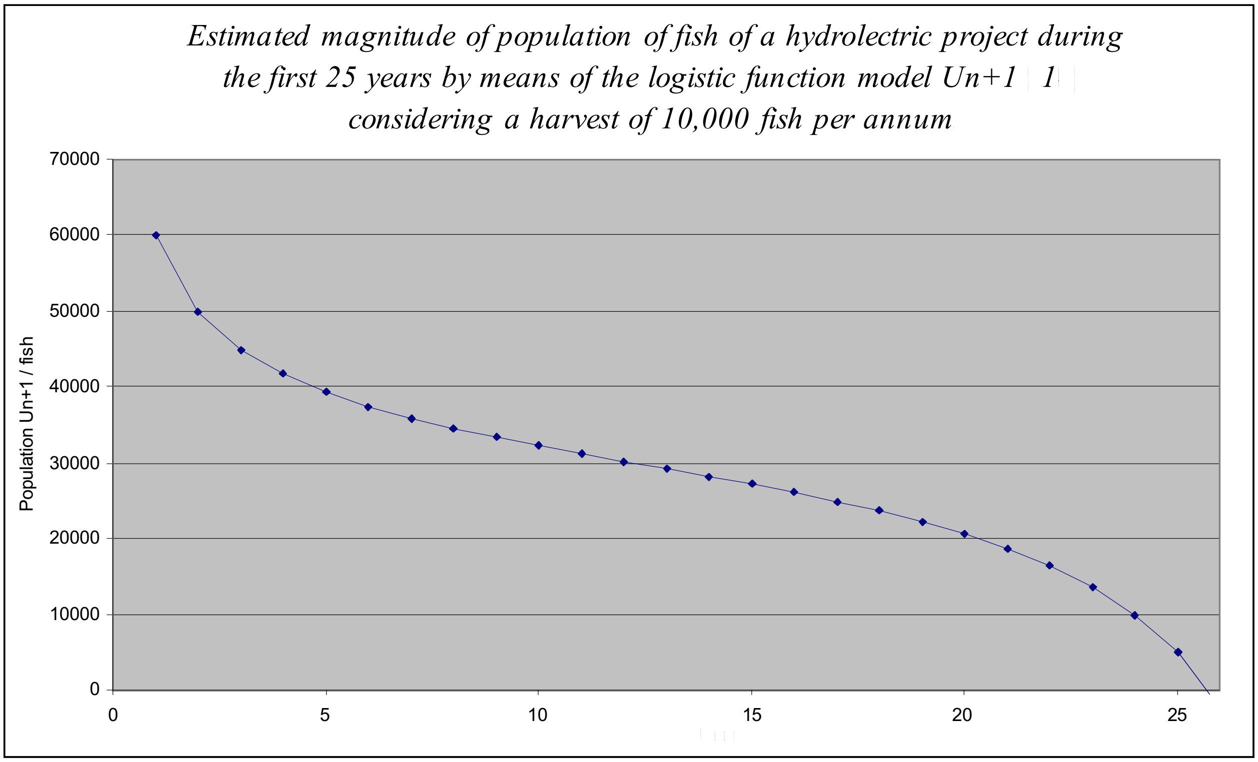

**Figure 7.2.** Graphical plot of the fish population of the hydroelectric project on an interval of 25 years using the logistic function model {14}. The model considers an annual harvest of 1x10$^4$ fish. A population collapse occurs after year 25.

From figure 7.2 one can observe the chronic depletion of the fish population as a consequence of excessive harvest. If one continues with the model into year 26 the population turns negative, and this means the death of the last fish in the lake. Thus it is **<u>not feasible</u>** to harvest above $1 \quad 10$ fish per annum, and one could impose a preliminary condition when harvesting H fish p.a.:

$$H \leq 1 \quad 10^{4}$$

8. <u>*Find the maximum annual sustainable harvest.*</u>

When harvesting and a stable population are attained, one could argue that the rate of harvesting equals the rate of growth of the population. But since one is not considering a differential equation, one could instead say that the population $u_{n+1}$ modelled in {12}, {13} and {14} can be generalized with model {15}:

$$u_{n+1} = (-1 \quad 10^{-5})(u_n \quad + 1.6(u_n) - H \qquad \{15\}$$

Because $u_1$=6.0x10$^4$ (a stable value) then r=1. Therefore it is valid to write:

$$u_{n+1} = (1)u$$

Thus if one considered $H=0$ one could write {3} as following

$$u_{n+1} = 1 \quad 10^{5} \quad {}_{+1} + 1 \quad )$$

Now, if one began harvesting H fish, the equation would look like the following:

$$u_{n+1} = (-1 \quad 10^{-5} \quad _{n+1})^2 + 1.6(u \quad) - H$$
$$)(u$$

$$® \lfloor (-1 \quad 10^{-})(u_{n+1})^2 + 1.6(u_{n+1})\rfloor - u_{n+1} - H = 0$$

$$® \ (-1 \quad 10^{-})(u_{n+1}) + 0.6(u_{n+1}) - H = 0 \qquad \{16\}$$

Model {16} is a quadratic equation which could be solved, if one knew the value of the constant term H, yet in this case it serves as an unknown that must be found. This could be solved by recurring to an analysis of the discriminant of the quadratic equation:

$$(-1 \quad 10^{-5}\ ( \quad ) + 0.6(u_{n+1}) - H = 0$$

$$® \ u_{n+1} = \frac{-0.6 \pm \sqrt{0.6^2 - 4(-1 \quad 10^{-})(-H)}}{2(-1 \quad 10^{-5})}$$

It is known that there can be

- ***a.*** Two real roots (D>0, $\sqrt{D} \in$ )
- ***b.*** One real root (D=0, $\sqrt{D} \in$ )
- ***c.*** No real roots (D<0, $\sqrt{D} \in$ **)**

For a solution (exact value for the maximum value for H) one could consider a single real root:

$$u_{n+1} = \frac{-0.6 \pm \sqrt{\cancel{0.6^2 - 4(-1 \quad 10^{-})(-H}})}{2(-1 \quad 10^{-5})} = \frac{-0.6 \pm \sqrt{D}}{2(-1 \quad 10^{-5})}$$

$$® \ D = 0.6^2 - 4(-1 \quad 10^{-})(-H) = 0$$

$$® \ H = -\left(\frac{-0.6^2}{-4(-1 \quad 10^{-5})}\right) = 9000 = 9 \quad 10$$

Thus, the maximum value of harvest, that still preserves the fish population at a stable value is ***H=9000***.

This value results in the following population development:

$$u_{n+1} = (-1 \quad 10^{-5})(u_{n+1})^2 + 1.6(u_{n+1}) - 9 \quad 10^3 \qquad \{17\}$$

**Table 8.1.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {14}. A harvest of $1x10^4$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 6.00 10 | 16 | 3. 9 10 |
| 2 | 5.10 10 | 17 | 3. 7 10 |
| 3 | .66 10 | 18 | 3. 5 10 |
|  | .38 10 | 19 | 3. 3 10 |
| 5 | .19 10 | 20 | 3. 1 10 |
| 6 | .05 10 | 21 | 3.39 10 |
| 7 | 3.9 10 | 22 | 3.38 10 |
| 8 | 3.85 10 | 23 | 3.36 10 |
| 9 | 3.78 10 | 2 | 3.35 10 |
| 10 | 3.72 10 | 25 | 3.3 10 |

| | | | |
|---|---|---|---|
| 11 | 3.67 10 | 26 | 3.33 10 |

| | | | |
|---|---|---|---|
| 12 | 3.62 10 | 27 | 3.32 10 |
| 13 | 3.58 10 | 28 | 3.30 10 |
| 1 | 3.55 10 | 29 | 3.30 10 |
| 15 | 3.52 10 | 30 | 3.30 10 |

The population ultimately reaches the value 3.00 10 .

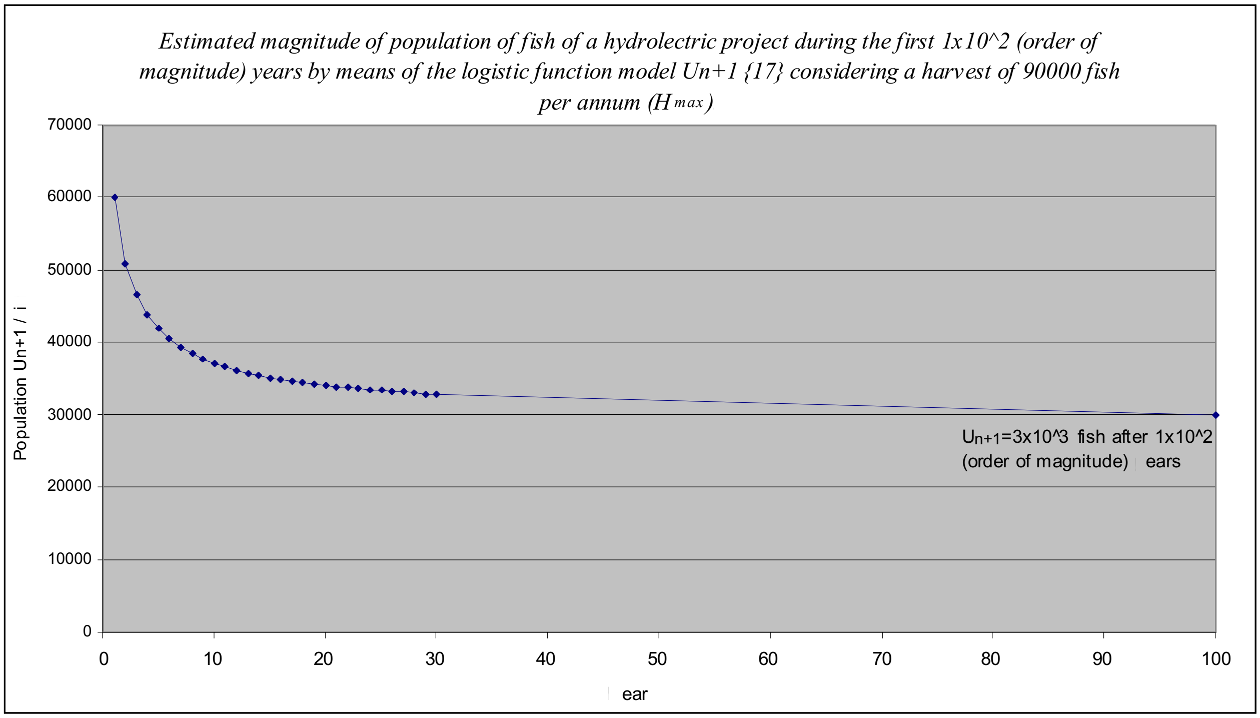

**Figure 8.1.** Graphical plot of the fish population of the hydroelectric project on an interval of ca. $1x10^2$ years (order of magnitude) using the logistic function model {17}. The model considers an annual harvest of $9x10^3$ fish. A stable population ($3x10^4$) is reached after over 100 years (thus the order of magnitude).

9. *<u>Politicians in the area are anxious to show economic benefits from this project and wish to begin the harvest before the fish population reaches its projected steady state. The biologist is called upon to determine how soon fish may be harvested after the initial introduction of 10,000 fish. Again using the first model in this task, investigate different initial population sizes from which a harvest of 8,000 fish is sustainable.</u>*

The population sizes that was investigated, was:

$$u_1 \le 2 \quad {}^{4}$$

$$10$$

This initial population is too small for it to be sustainable:

**Table 9.0.** The population of fish in a lake over a time range of 20 years estimated using the logistic function model {14}. A harvest of 8x10$^{3}$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 1.95 10 | 11 | 1.65 10 |
| 2 | 1.9 10 | 12 | 1.57 10 |
| 3 | 1.93 10 | 13 | 1. 7 10 |
| | 1.91 10 | 1 | 1.33 10 |
| 5 | 1.89 10 | 15 | 1.16 10 |
| 6 | 1.87 10 | 16 | 9.16 10$^{3}$ |
| 7 | 1.8 10 | 17 | 5.82 10$^{3}$ |
| 8 | 1.81 10 | 18 | 9.68 10$^{2}$ |
| 9 | 1.77 10 | 19 | -6. 6 10$^{3}$ |
| 10 | 1.72 10 | 20 | -1.88 10 |

Thus one can say, that the minimum initial population must be

$$u_1 \geq 2 \quad 10^{4}$$

***a.*** Consider an initial population $u_1$=2.5x10$^{4}$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1 \quad 10^{-5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8 \quad 10^{3} \quad ) \qquad \{19\}$$

**Table 9.1.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {14}. A harvest of 1x10$^{4}$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 2.5 10 | 16 | 3.75 10 |
| 2 | 2.58 10 | 17 | 3.80 10 |
| 3 | 2.66 10 | 18 | 3.83 10 |
| | 2.75 10 | 19 | 3.86 10 |
| 5 | 2.8 10 | 20 | 3.89 10 |
| 6 | 2.9 10 | 21 | 3.91 10 |
| 7 | 3.0 10 | 22 | 3.93 10 |
| 8 | 3.1 10 | 23 | 3.9 10 |
| 9 | 3.23 10 | 2 | 3.95 10 |
| 10 | 3.33 10 | 25 | 3.96 10 |
| 11 | 3. 2 10 | 26 | 3.97 10 |
| 12 | 3.50 10 | 27 | 3.98 10 |
| 13 | 3.58 10 | 28 | 3.98 10 |
| 1 | 3.6 10 | 29 | 3.98 10 |
| 15 | 3.70 10 | 106 | .00 10 |

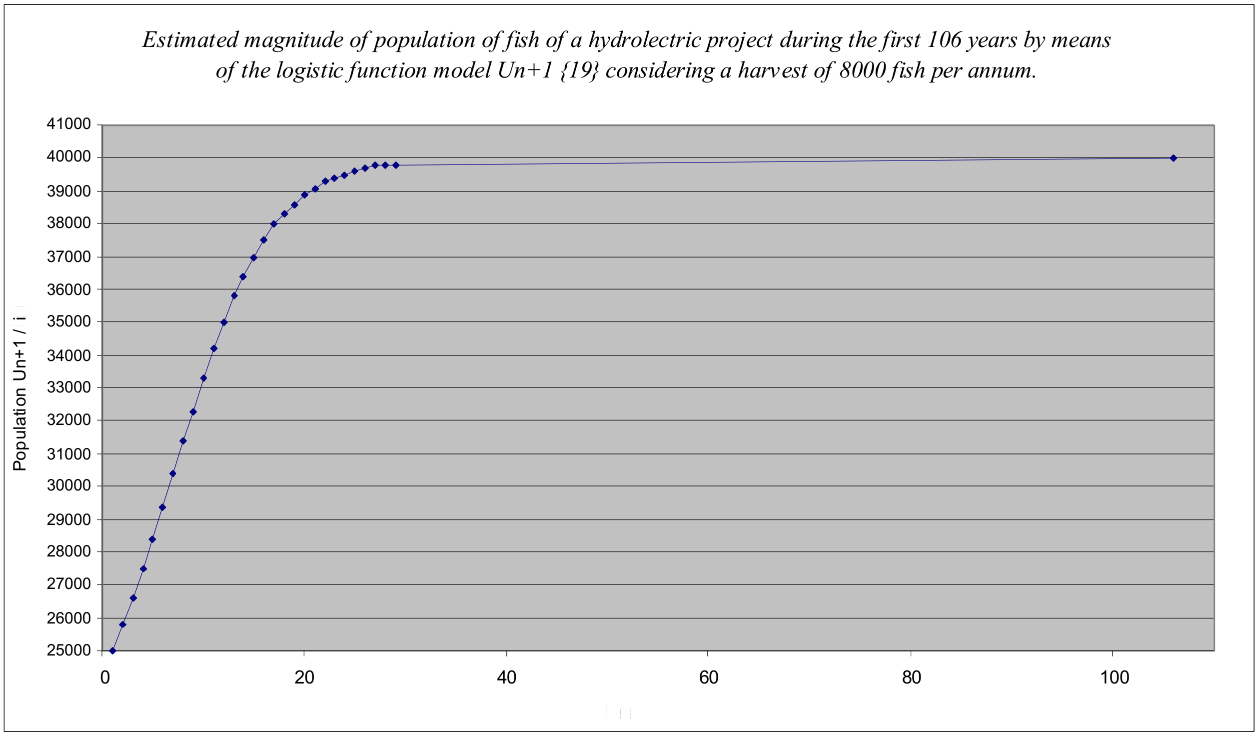

**Figure 9.1.** Graphical plot of the fish population of the hydroelectric project on an interval of 106 years using the logistic function model {19}. The model considers an annual harvest of 8x$10^3$ fish. A stable population (4x$10^4$) is reached.

***b.*** Consider an initial population $u_1$=3.0x$10^4$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1\ 10^{5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8\ 10^{3} \qquad \{19\}$$

**Table 9.2.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {19}. A harvest of 8x$10^3$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 3.00 10 | 16 | 3.9 10 |
| 2 | 3.20 10 | 17 | 3.95 10 |
| 3 | 3.30 10 | 18 | 3.96 10 |
|  | 3.39 10 | 19 | 3.97 10 |
| 5 | 3. 7 10 | 20 | 3.97 10 |
| 6 | 3.55 10 | 21 | 3.98 10 |
| 7 | 3.62 10 | 22 | 3.98 10 |
| 8 | 3.68 10 | 23 | 3.99 10 |
| 9 | 3.73 10 | 2 | 3.99 10 |
| 10 | 3.78 10 | 25 | 3.99 10 |
| 11 | 3.82 10 | 26 | 3.99 10 |
| 12 | 3.85 10 | 27 | 3.99 10 |
| 13 | 3.88 10 | 28 | .00 10 |
| 1 | 3.90 10 | 29 | .00 10 |

| | | | |
|---|---|---|---|
| 15 | 3.92 10 | 30 | .00 10 |

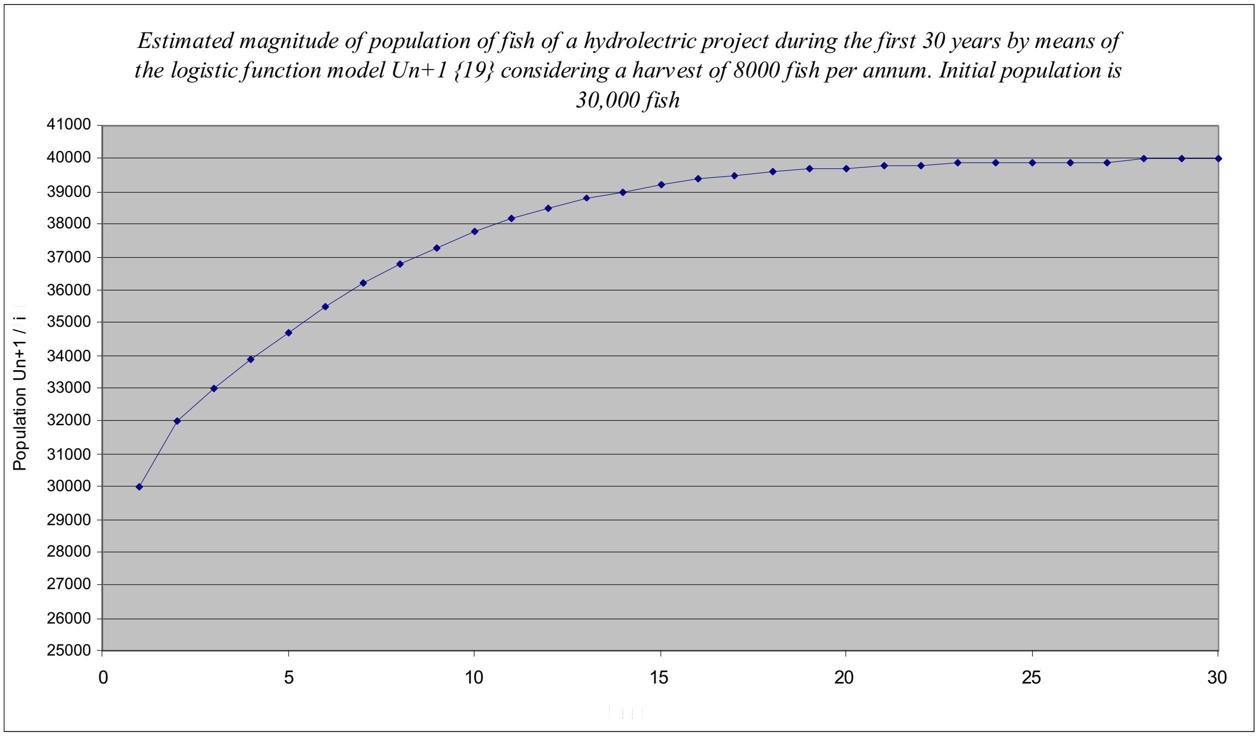

**Figure 9.2.** Graphical plot of the fish population of the hydroelectric project on an interval of 30 years using the logistic function model {19}. The model considers an annual harvest of 8x$10^3$ fish. A stable population (4x$10^4$) is reached.

***c.*** Consider an initial population $u_1$=3.5x$10^4$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1\ \ 10^{-5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8\ \ 10^{3} \qquad \{19\}$$

**Table 9.3.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {19}. A harvest of 8x$10^3$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 3.50 10 | 16 | 3.98 10 |
| 2 | 3.58 10 | 17 | 3.98 10 |
| 3 | 3.6 10 | 18 | 3.98 10 |
| | 3.70 10 | 19 | 3.99 10 |
| 5 | 3.75 10 | 20 | 3.99 10 |
| 6 | 3.80 10 | 21 | 3.99 10 |
| 7 | 3.83 10 | 22 | 3.99 10 |
| 8 | 3.86 10 | 23 | 3.99 10 |
| 9 | 3.89 10 | 2 | 3.99 10 |
| 10 | 3.91 10 | 25 | .00 10 |
| 11 | 3.93 10 | 26 | .00 10 |

| | | | |
|---|---|---|---|
| 12 | 3.94 10 | 27 | .00 10 |
| 13 | 3.95 10 | 28 | .00 10 |
| 1 | 3.96 10 | 29 | .00 10 |
| 15 | 3.97 10 | 30 | .00 10 |

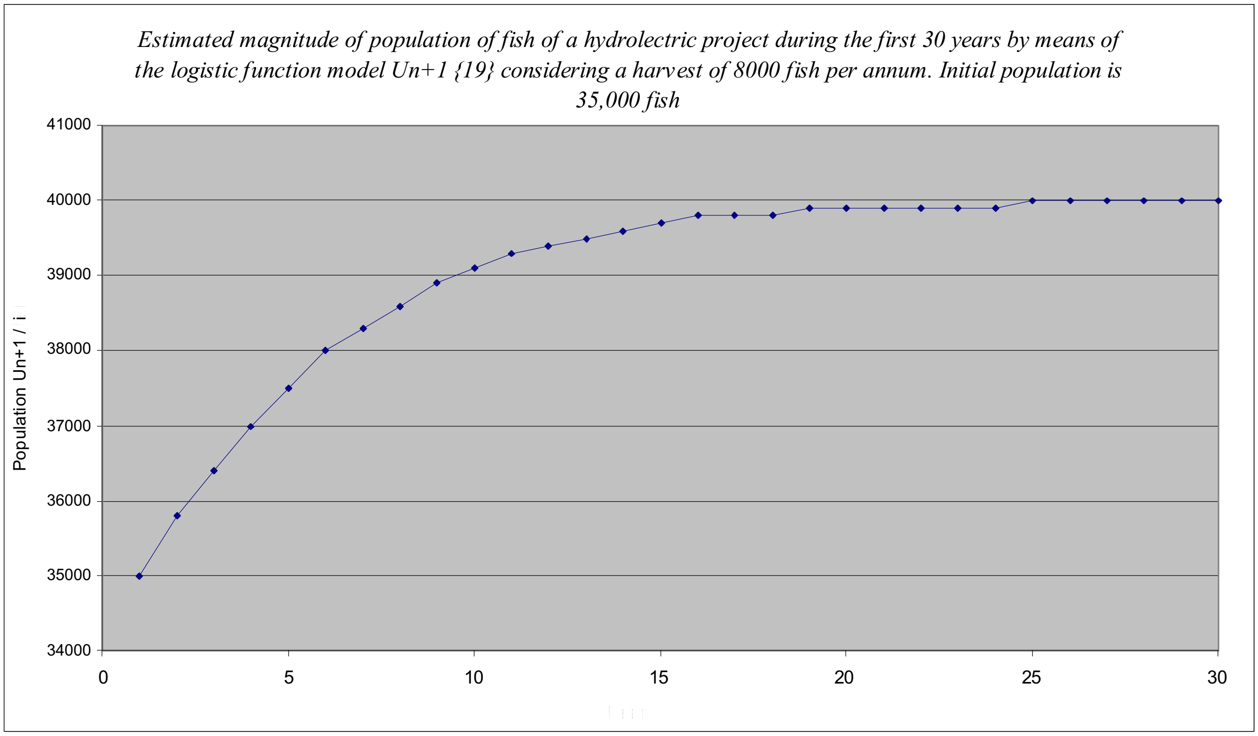

**Figure 9.3.** Graphical plot of the fish population of the hydroelectric project on an interval of 106 years using the logistic function model {19}. The model considers an annual harvest of 8x$10^3$ fish. A stable population (4x$10^4$) is reached.

***d.*** Consider an initial population $u_1$=4.0x$10^4$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1 \quad 10^{-5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8 \quad 10^{3} \qquad \{19\}$$

**Table 9.4.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {19}. A harvest of 8x$10^3$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 4.00 10 | 16 | .00 10 |
| 2 | .00 10 | 17 | .00 10 |
| 3 | .00 10 | 18 | .00 10 |
| | .00 10 | 19 | .00 10 |
| 5 | .00 10 | 20 | .00 10 |
| 6 | .00 10 | 21 | .00 10 |
| 7 | .00 10 | 22 | .00 10 |
| 8 | .00 10 | 23 | .00 10 |

| | | | |
|---|---|---|---|
| 9 | 4.00 10 | 2 | .00 10 |
| 10 | .00 10 | 25 | .00 10 |
| 11 | .00 10 | 26 | .00 10 |
| 12 | .00 10 | 27 | .00 10 |
| 13 | .00 10 | 28 | .00 10 |
| 1 | .00 10 | 29 | .00 10 |
| 15 | .00 10 | 30 | .00 10 |

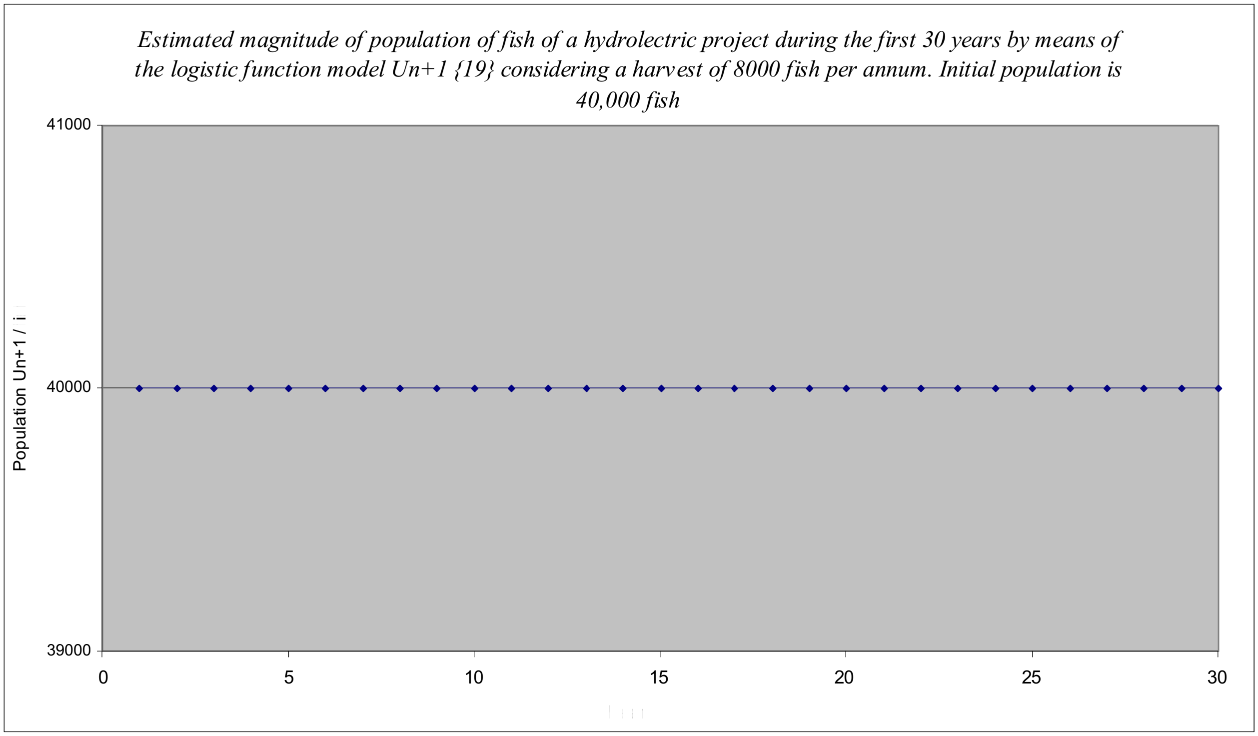

**Figure 9.4.** Graphical plot of the fish population of the hydroelectric project on an interval of 30 years using the logistic function model {19}. The model considers an annual harvest of 8x$10^3$ fish. A stable population (4x$10^4$) is reached.

***e.*** Consider an initial population $u_1$=4.5x$10^4$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1 \quad 10^{-5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8 \quad 10^{3} \qquad \{19\}$$

**Table 9.5.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {19}. A harvest of 8x$10^3$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 4.00 10 | 16 | .01 10 |
| 2 | .38 10 | 17 | .01 10 |
| 3 | .29 10 | 18 | .01 10 |
| | .22 10 | 19 | .01 10 |

| | | | |
|---|---|---|---|
| 5 | 4.17 10 | 20 | .01 10 |
| 6 | .13 10 | 21 | .00 10 |
| 7 | .11 10 | 22 | .00 10 |
| 8 | .08 10 | 23 | .00 10 |
| 9 | .07 10 | 2 | .00 10 |
| 10 | .05 10 | 25 | .00 10 |
| 11 | .0 10 | 26 | .00 10 |
| 12 | .03 10 | 27 | .00 10 |
| 13 | .03 10 | 28 | .00 10 |
| 1 | .02 10 | 29 | .00 10 |
| 15 | .02 10 | 30 | .00 10 |

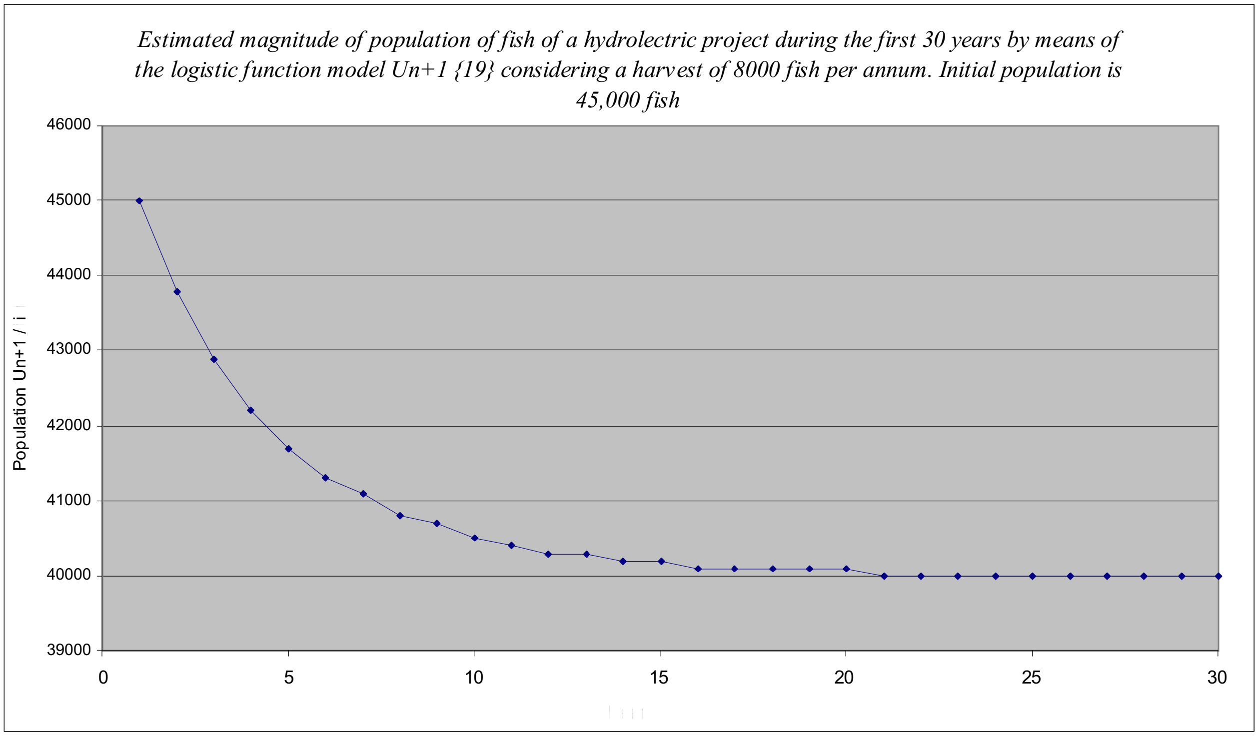

**Figure 9.5.** Graphical plot of the fish population of the hydroelectric project on an interval of 30 years using the logistic function model {19}. The model considers an annual harvest of $8x10^3$ fish. A stable population ($4x10^4$) is reached.

***f.*** Consider an initial population $u_1$=$6.0x10^4$ fish.

The equation could be formulated as follows:

$$u_{n+1} = (-1 \quad 10^{-5})(u_{n+1}^{2} + 1.6(u_{n+1}) - 8 \quad 10^{3} \qquad \{19\}$$

**Table 9.6.** The population of fish in a lake over a time range of 30 years estimated using the logistic function model {19}. A harvest of $8x10^3$ fish is made per annum.

| Year (after attaining stability) | Population | Year | Population |
|---|---|---|---|
| 1 | 6.00 10 | 16 | .03 10 |
| 2 | 5.20 10 | 17 | .02 10 |
| 3 | .82 10 | 18 | .02 10 |
|  | .59 10 | 19 | .02 10 |
| 5 | . 3 10 | 20 | .01 10 |
| 6 | .33 10 | 21 | .01 10 |
| 7 | .25 10 | 22 | .01 10 |
| 8 | .20 10 | 23 | .01 10 |
| 9 | .15 10 | 2 | .00 10 |
| 10 | .12 10 | 25 | .00 10 |
| 11 | .09 10 | 26 | .00 10 |
| 12 | .07 10 | 27 | .00 10 |
| 13 | .06 10 | 28 | .00 10 |
| 1 | .05 10 | 29 | .00 10 |
| 15 | .0 10 | 30 | .00 10 |

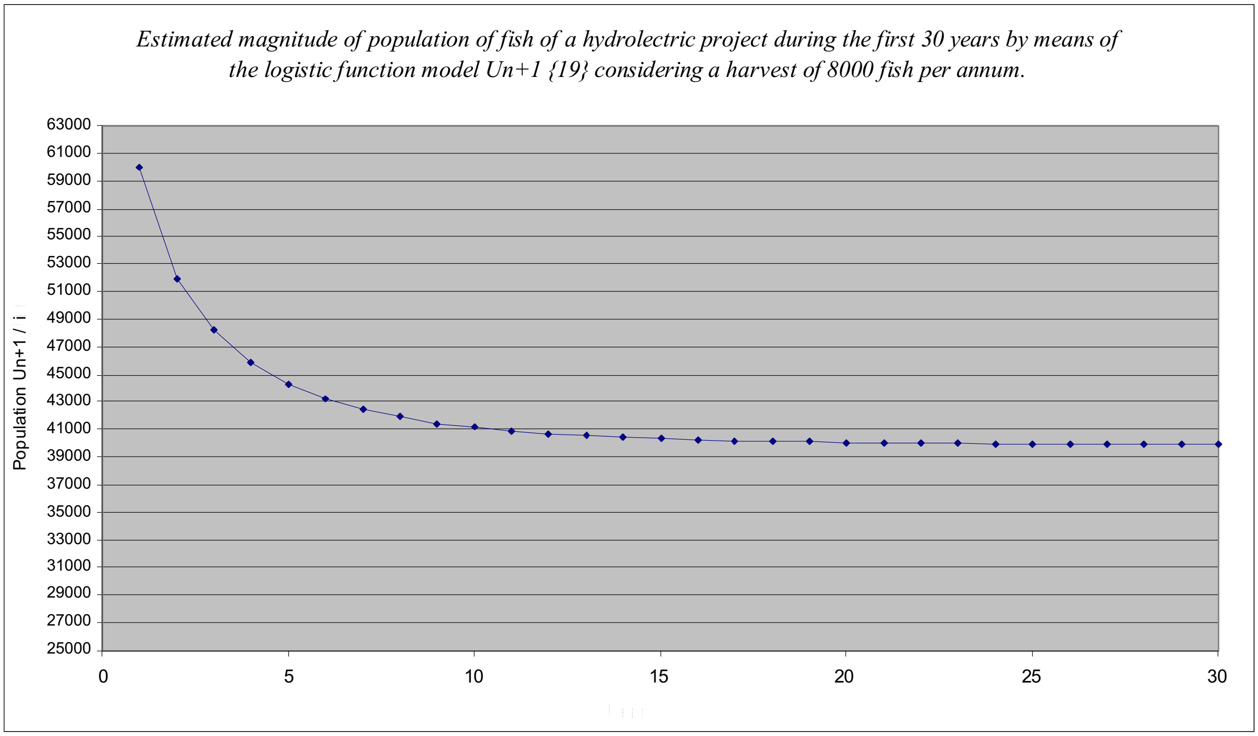

**Figure 9.6.** Graphical plot of the fish population of the hydroelectric project on an interval of 30 years using the logistic function model {19}. The model considers an annual harvest of $8x10^3$ fish. A stable population ($4x10^4$) is reached.

One can conclude that the population reaches a positive, stable limit in the following interval:

$$u_{n+1} \in [2 \quad 10^4, 1.4 \quad 10 [$$

Also, as $u_{n+1} \to 1.4 \quad 10$ the stable population instead of tending (and ultimately becoming) $4 \quad 10^4$, it ,

tends towards $2 \quad 10^4$, and when $u_{n+1} = 1.4 \quad 10^5$, the stable population is exactly $2 \quad 10^4$.
Thus it can be concluded, that when the initial population exceeds the stable limit, it will decrease in magnitude, until reaching $4 \quad 10$ fish, and if it lies below the stable population limit ( $\quad 10$ ) it will increase in magnitude until reaching stability. If one was to plot one initial population size of $u_1 < 4 \quad 10^4$ versus $u_1 > 4 \quad 10$ an **equilibrium** (resembling the chemical equilibrium of a reversible reaction) is attained:

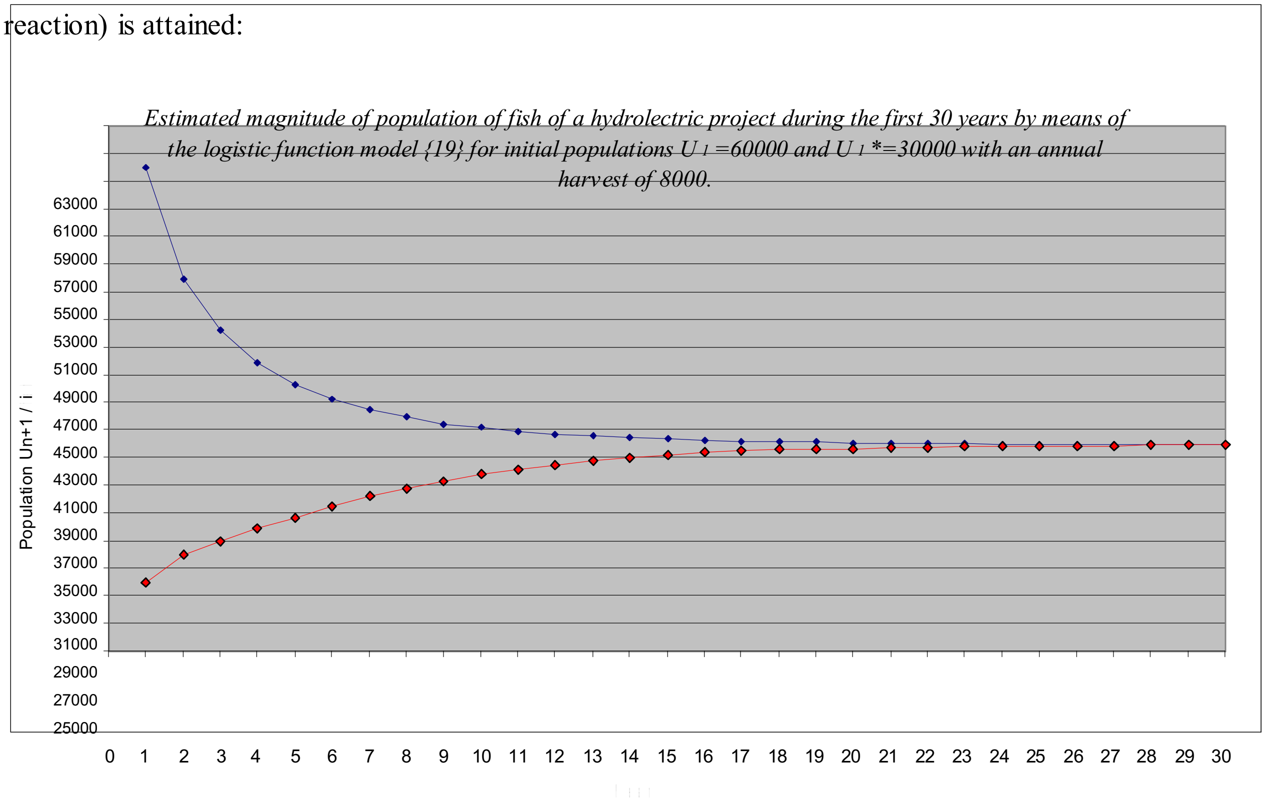

**Figure 9.7.** Two initial population sizes attaining equilibrium at $10^4$ for the model {19}.

What is the usefulness of such equilibrium? It tells us that if the politicians wanted to start such a project with a set annual harvest, they could do this **at the lowest cost** by starting with a population of $3\text{x}10^4$ fish instead of $6\text{x}10^4$ fish, **obtain the same yearly harvest** and keep the population alive. This is the major socio-economic benefit of this study.

Another point to this analysis is that, if one started with either of following values:

$$u_1 = 2 \quad 10^4$$
$$u_1 = 1.2 \quad 10^5$$
$$u_1 = 1.4 \quad 10^5$$

One would get to the same stable limit of $2 \quad 10$ :

$$u_{n+1} = (-1 \quad 10^{-5})(2 \quad 10^4)^2 + 1.6(2 \quad -8 \quad 10 \quad 10^4)$$

= 2

10

$$u_{n+1} = (-1 \times 10^{-5})(1.2 \times 10^{5})^{2} + 1.6(1.2 \times 10^{5}) - 8 \times 10^{3}$$
$$= 2 \times 10$$

$$u_{n+1} = (-1 \times 10^{-5})(1.4 \times 10^{5})^{2} + 1.6(1.4 \times 10^{5}) - 8 \times 10^{3}$$
$$= 2 \times 10^{4}$$

When graphing this one gets the follow figure:

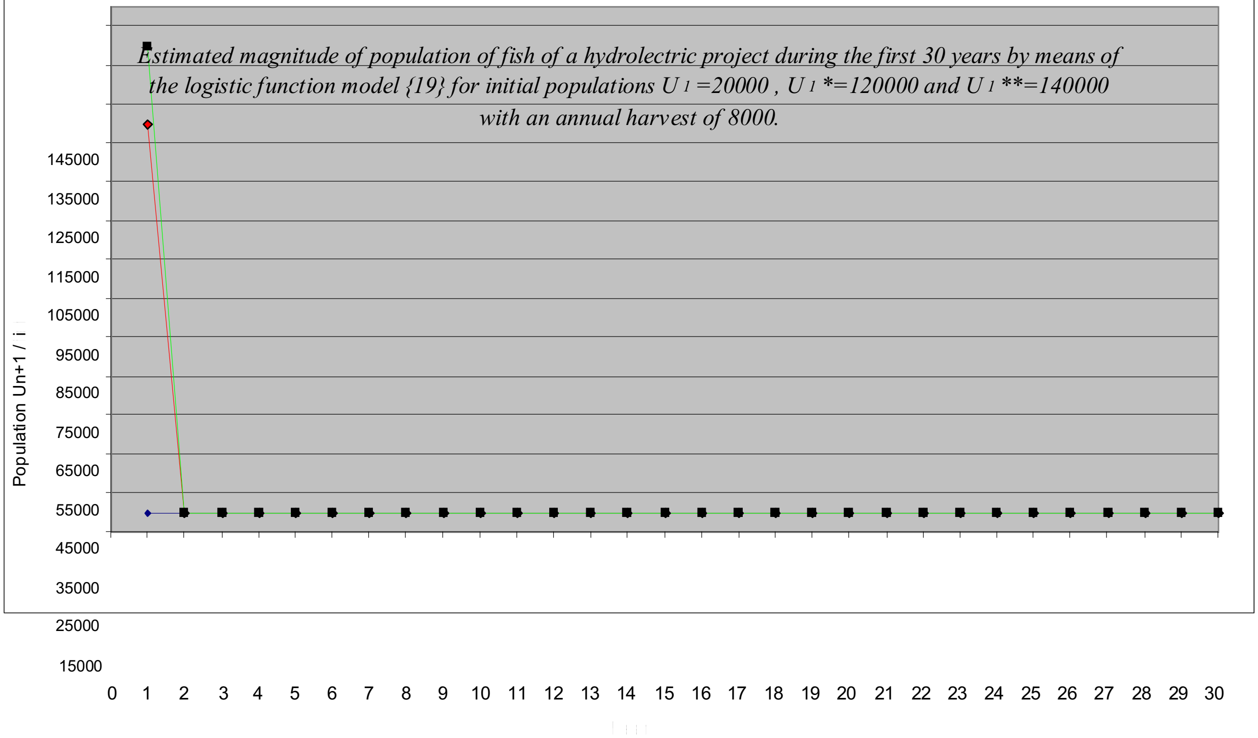

**Figure 9.8.** Three initial population sizes developing throughout 30 years time according to the model {19}.

Thus by starting the project with 2x10$^{4}$ one could (theoretically) maintain a stable population and get a fairly high harvest out of it.

## Conclusion

The logistic function {3} has been applied throughout the task and conclusions can be made about the limitations of the model.

The model is strictly based on the assumption that r will remain constant at a value of r=1.5 until the stable population limit is reached. Thus were the population to be influenced by some external factor, the model would not take this into account and the growth of the population would undergo a discrepancy.

One could argue that the following assumptions were made when constructing the model:

- The fish are ovoviviparous and carry the egg until it hatches[1].
- The fish population is somewhat homogenous.

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[1] Fish reproduction. < http://lookd.com/fish/reproduction.html> (online). Visited on 21. December 2006

- Their rate of reproduction is simplified and expressed as a growth factor instead of a rate of change

Two cases could occur:

a. An external factor increases the mortality of the population.
b. The fish reproduced at a higher rate than the model predicts.

Also, one has not taken into account that despite the model having the long-term sustainability limit at *about* 6x10$^4$, several factors could change this:

- Human action such as contamination of the habitat.
- Other aquatic vertebrates[2] that might be positioned higher in the food chain, and could disturb the stability by consuming the studied fish's own source of food.
- Season changes
- Global warming

Thus it would be wise to introduce a term into the model that takes into the account the long-term sustainability limit, which in a non-ideal situation probably would change.

The model predicts a few interesting things:

- Overpopulating the lake in the first year leads to the death of all fish by the end of year one. The model is built for $u_1$=1x10$^4$ fish, yet say one introduced one million fish in the lake (ignoring the capacity of the lake):

$$u_{1+1} = (-1\quad 10^{-5})(u_1)^2 + 1.6(u_1)$$
$$= (-1\quad 10^{-5})(1\quad 10^6)^2 + 1.6(1\quad 10^6) = -8.4\quad 10^6\ fish$$

- Populations that reproduce at a very high rate (ie. r=2.9) result in a dynamic limit (with two values instead of a stable limit with only one value) of fluctuations between the extrema, first from the maximum towards the minimum in only one year. This opposes the asymptotic behaviour predicted when the growth factor is smaller.
- When harvesting is introduced into a stable population (in this case at $u_{1=6x10^4}$) only harvests below 9x10$^3$ fish may take place unless extinction of the population is desired.
- When the harvesting is set at constant value of 8x10$^3$, initial populations below a specific value will grow in magnitude until approaching that value, while initial populations above that value (yet not above a certain limit as overpopulation causes death as well) lead to a decrease in magnitude until approaching the same limit.

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[2] Biological term for fish: Wikipedia < http://en.wikipedia.org/wiki/Fish> Visited on 21. December 2006