

# The History of the Cubic Equation

*Biologists think they are molecular biologists. Molecular biologists think they are chemists.  
Chemists think they are physical chemists. Physical chemists think they are physicists.  
Physicists think they are God.  
God thinks He is a mathematician! Author Unknown*

(a)(i) Scipione del Ferro (ca. 1465 – 1526) is the first person known to have solved cubic equations algebraically. However, he did not disclose this remarkable achievement until he was near death. The momentous solution fell to the ears of Antonio Maria Fior, a student of del Ferro, who, upon gaining the information, began to boast that he was able to solve cubic equations. This delusion of grandeur gave rise to a public problem-solving contest between Fior and Niccolo Fontana or Tartaglia as he was more widely known. It was at this contest, set in Venice on the 22 February 1535 that Fior posed problem  $\alpha$  to Tartaglia.

*( $\alpha$ ) There was a tree of height 12, which was broken in two in such a way that the height of the part remaining was the cube root of the part cut away. What was the height of the part remaining?*

Fortunately for the latter, a moment of inspiration had come ten days previously. Tartaglia had discovered a method of solving two types of cubic, those of the form *a cube and things equal to numbers* ( $x^3 + px = q$ ) and *a cube and squares equal to numbers* ( $x^3 + px^2 = q$ ). Fior could only solve equations in the first form (del Ferro's type) and so the solution of other type set by his opponent eluded him. Tartaglia, armed with more tools reduced his challenges into special cases of the cubic  $x^3 + px = q$  and quickly solved all thirty of Fior's problems in less than two hours.

In 1548 Tartaglia became involved in another public contest, this time his opponent was Ludovico Ferrari. It was to be held in Milan on the 10<sup>th</sup> August at the church garden of the Fratti Zoccolanti in front of a large, distinguished audience. It was here that the following problem was posed.

*( $\beta$ ) Divide 8 into two parts such that their product multiplied by their difference comes to as much as possible.*

- (a)(ii) (α) *There was a tree of height 12, which was broken in two in such a way that the height of the part remaining was the cube root of the part cut away. What was the height of the part remaining?*

This problem translates to the cubic equation

$$x^3 + x = 12 . \quad (\clubsuit)$$

A formula, developed by Cardan can be used to solve a cubic in this form. His formula states that for a cubic equation in the form  $x^3 + px = q$ ,

$$x = \sqrt[3]{\sqrt{\left\{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2\right\} + \frac{q}{2}}} - \sqrt[3]{\sqrt{\left\{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2\right\} - \frac{q}{2}}}$$

However, this method will only give one value for  $x$ . In order to solve for all values of  $x$  another formula must be employed. Firstly it is necessary to determine the nature of the roots of the quadratic. Equation ( $\clubsuit$ ) has one real root and a pair of complex roots and these roots can be determined by the formulas

$$\begin{aligned} X_1 &= (S + U) - \left(\frac{b}{3a}\right) \\ X_2 &= -\frac{(S + U)}{2} - \left(\frac{b}{3a}\right) + i(S - U)\frac{\sqrt{3}}{2} \\ X_3 &= -\frac{(S + U)}{2} - \left(\frac{b}{3a}\right) - i(S - U)\frac{\sqrt{3}}{2} \end{aligned} \quad (\spadesuit)$$

where each of the variables are determined by

$$\begin{aligned} S &= \sqrt[3]{R}, & U &= \sqrt[3]{T}, \\ R &= -\left(\frac{g}{2}\right) + \sqrt{h}R = -\left(\frac{g}{2}\right) + \sqrt{h}, & T &= -\left(\frac{g}{2}\right) - \sqrt{h}, \\ g &= \frac{2b^3/a^3 - 9bx/a^2 + 27d/a}{27}, & h &= \left(\frac{g^2}{4}\right) + \left(\frac{f^3}{27}\right). \end{aligned}$$

a, b, c and d correspond to real numbers in the general quadratic

$$ax^3 + bx^2 + cx + d = 0$$

Fortunately equation (♣) is a little simpler than the above form because it has no  $x^2$  term, this implies that  $b = 0$ . If the number 12 is taken away from both sides to give  $d = -12$  then, together with the previous two, the numbers  $a = 1$  and  $c = 1$  can be substituted into the above formulas to obtain

$$f = 1, \quad g = -12, \quad h = \frac{9B}{2},$$

$$R = 6 + \sqrt{\frac{9B}{2}}, \quad T = 6 - \sqrt{\frac{9B}{2}},$$

$$S = \sqrt[3]{6 + \sqrt{\frac{9B}{2}}}, \quad U = \sqrt[3]{6 - \sqrt{\frac{9B}{2}}}.$$

Now these numbers can be substituted into equations (♠) and the following values are obtained for the roots of the cubic polynomial:

$$X_1 = \sqrt[3]{6 + \sqrt{\frac{9B}{2}}} + \sqrt[3]{6 - \sqrt{\frac{9B}{2}}} = 2.140062 \quad \dots$$

$$X_2 = -1.07003 \quad \dots + i2.185827 \quad \dots$$

$$X_3 = -1.07003 \quad \dots - i2.185827 \quad \dots$$

These results are likely to be quite different from the ones originally given. Firstly Tartaglia would only have looked for one value  $x$  and only found one value of  $x$ . In fact roots of negative numbers never occurred to Tartaglia and were not understood by Cardan. Tartaglia would also have used a different method of solution and would not have given his solution in the above algebraic form.

(β) Divide 8 into two parts such that their product multiplied by their difference comes to as much as possible.

Dividing eight into two parts corresponds to the equation

$$x + y = 8 \quad (1)$$

Now, if the product of these numbers,  $x$  and  $y$ , multiplied by their difference must come to as much as possible it is necessary that the following equation is satisfied

$$xy(x - y) = M, \quad (2)$$

where  $M$  denotes maximum value

If equation (1) is rearranged then the following equation is obtained

$$y = 8 - x \quad (3)$$

Now equation (3) can be substituted into equation (2) to obtain

$$x = x(8 - x)(x - [8 - x]) = M$$

$$\Rightarrow (8x - x^2)(2x - 8) = M$$

$$\Rightarrow 24x^2 - 64x - 2x^3 = M$$

We have now obtained a function of  $x$ . In order to find the values of  $x$  at a maximum point the  $f(x)$  obtained must be differentiated and solved equal to zero, i.e.

$$f'(x) = -6x^2 + 48x - 64$$

$$f'(x) = 0 \Rightarrow 6x^2 - 48x + 64 = 0$$

$$\Rightarrow x^2 - 8x + \frac{32}{3} = 0$$

Using the formula for solving quadratics we find

$$\begin{aligned} x &= \frac{8 \pm \sqrt{64 - 4.1(32/3)}}{2} \\ &= 4 \pm \frac{4}{\sqrt{3}} \end{aligned}$$

So the maximum value for  $x$  is

$$x = 6.399077 \dots$$

Substituting into (3) gives

$$y = 1.0082 \dots$$

The value found for  $y$  is actually the same as the alternative minimum value for  $x$ . Again, the method of solution used here would be very different from that used in renaissance Italy. As with problem ( $\alpha$ ) there would not have been the same style of algebra. Indeed, there would also have been no notion of Calculus. The idea here of maxima and minima would have to wait over one hundred years to be formalised.

(b) *What can be learned from the study of the history of the cubic equation about mathematics / mathematical life in sixteenth century Italy?*

## 1) Secrecy of Discovery and Publishing

The story of the history of the cubic equation illustrates a change in mathematics that came with the Renaissance: the understanding that mathematicians must publish their work to succeed; that instead of keeping mathematical knowledge as secrets within families, to be used as weapons in debates, or in mathematics contests, they must be available to the public. The solution of the cubic equation came at a time when both these conflicting views on mathematics were both in common practice; furthermore, the story involved controversy between mathematicians from both sides of the argument, which seem to reflect perfectly the changing ideas of how mathematics should be practiced. This is exemplified by the controversy between the two names most commonly associated with the “cubic formula,” or the solution to the cubic equation: Gerolamo Cardano (1501-1576), the prestigious physician, astrologer and mathematician who first published the result, and Niccolo Fontana, known as Tartaglia (1506-1557), a self-educated mathematician who secretly derived the formula to use to his advantage in a mathematics contest, and was outraged when Cardano published it without permission. Cardano’s actions were representative of the new attitudes of Renaissance mathematicians—for instance, that publishing should be synonymous with prestige— while Tartaglia represents the underdeveloped philosophy of the middle ages, that mathematics should be kept secret from the public so that mathematicians may gain personal prestige. Cardano acted appropriately as a “Renaissance man,” and made the decision that was aimed toward the greater good of mathematics. Cardano heard about Tartaglia and his solution to the cubic equation, and wrote a letter asking Tartaglia to share his solution so that he might publish in his next book on algebra, *Ars Magna*. Tartaglia responded denying Cardanos’ request and proposed that *his* result would only see the light of day when *he* chose to publish a work of his own. However, Cardano eventually managed to persuade Tartaglia to share his secret, on the condition that he took an oath never to divulge it to anybody. Six years later, Cardano published *Ars Magna*, which included Tartaglias' formula. From one point of view, one could conclude that Cardano was merely acting on the new philosophies of the Renaissance, and that, in complaining about the

publishing of his formula, Tartaglia was merely clinging to the medieval views about mathematics, which were much less efficient in producing mathematics, and even more unfair in assigning credit where it was due. Some of the trends of keeping mathematical advancements out of the public eye were still occurring during the Italian Renaissance. Until the printing press became popular in the mid-sixteenth century, it was common practice to keep mathematical knowledge a secret within a mathematician's family, and, at the most, to share it with one's apprentices. The primary reason is that mathematical contests, such as the contest between Tartaglia and Fiore, were still common practice among mathematical scholars. Mathematical knowledge could be a powerful weapon until it was exposed to the public.

## 2) Complex Numbers

Four and a half centuries have elapsed since the discovery of complex, or 'imaginary' numbers, a discovery which ultimately has had a profound impact on the whole of mathematics, unifying much that had previously seemed disparate and explaining much that had previously been inexplicable. Far from being embraced, complex numbers were initially greeted with suspicion, confusion and even hostility. Cardano's *Ars Magna* (1545) is traditionally taken to be the birth certificate of complex numbers. It may seem a little strange therefore that he introduced them only to dismiss them with the comment '*they are as subtle as they are useless*'. Building on the work of del Ferro and Tartaglia, Cardano's *Ars Magna* showed cubic equations could be solved by the remarkable formula

$$x = \sqrt[3]{p + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} .$$

When considering the problem  $x^3 = 15x + 4$  Cardano encountered a solution which required the square root of  $-121$ , i.e. a negative number. Such a solution was called a *casus irreducibilis*. Cardano was aware that negative numbers did not possess square roots but he also knew that  $x=4$  was a solution to the above equation. "*He was not able to make sense of this paradox; he had, as it were, blundered into the minefield of complex numbers.*" <sup>(Hollingdale)</sup> Clearly the challenge presented by the existence of

complex numbers could not be ignored much longer. Bombelli was the first mathematician bold enough to accept the existence of imaginary numbers, and hence to throw some light on the puzzle of irreducible cubic equations. While complex numbers themselves remained mysterious, Bombelli's work on cubic equations established that real problems required complex arithmetic for their solution. "*In proving the reality of the roots of the cubic  $x^3 = 15x + 4$ , Bombelli demonstrated the extraordinary fact that real numbers could be engendered by imaginary numbers.*" <sup>(Burton)</sup> It was because of the study and development of the cubic equation that imaginary numbers lost some of their mystical character. However, just as with the birth, the subsequent development of the theory was inextricably bound with the progress in other areas of mathematics.

### 3) The Battle of the Scholars – Public Contests

At the turn of the sixteenth century in Italy, a teacher of mathematics lived in a highly competitive world. At this time, students paid their professors directly for each course they took. Thus, if they became dissatisfied with the level or quality of instruction, payment could be summarily suspended, and the instructor could be forced to leave the school and even the town. To uphold their reputations and to insure their livelihoods, professors engaged in public contests, which were commonplace among mathematical scholars. The contests had a huge influence on scholars, first because the prize was often a considerable sum of money, and second because positions at universities were not tenured, and maintaining professorships depended heavily on one's performance in contests. These contests were generally initiated by an underdog who proposed a series of problems to an established figure. The better-known mathematician then prepared a comparable set of examples for the challenger. After a predetermined length of time, the participants came together in public to present their solutions, the one with the greater number of correct answers taking the contest. In such an atmosphere, the guardian of a new solution or technique gained a distinct advantage over potential opponents and enjoyed job security by virtue of his secret. Given the system, it was simply not in one's best interests to publicize major discoveries. Tartaglia was a strong proponent of this attitude towards secrecy. In addition to his contest with Fiore, Tartaglia participated in a large number of public



contests, in which he was quite successful. In fact, he made a large part of his living from the prize money he received from public disputes and contests.

Indeed, the study of the cubic equation certainly helped to revive the zest and vigour which early day mathematicians experienced. Let it be said that Italy has spawned some of the world's greatest artists, for in the renaissance, Italy saw the development of some of histories most magical artists – determined pure mathematicians.

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