

The Fibonacci Sequence and Generalizations

Abstract: This paper gives a brief introduction to the famous Fibonacci sequence and demonstrates the close link between matrices and Fibonacci numbers. The much-studied Fibonacci sequence is defined recursively by the equation $y_{k+2} = y_{k+1} + y_k$, where $y_1 = 1$ and $y_2 = 1$. By using algebraic properties of matrices, we derive an explicit formula for the k th Fibonacci number as a function of k and an approximation for the “golden ratio” y_{k+1} / y_k . We also demonstrate how useful eigenvectors and eigenvalues can be in understanding the dynamics of linear recurrence relations of the form $y_{k+2} = ay_{k+1} + by_k$ where $a, b \in \mathbb{R}$.

I. Introduction

The Fibonacci sequence, probably one of the oldest and most famous sequences of integers, has fascinated both amateur and professional mathematicians for centuries. Named after its originator, Leonardo Fibonacci, the Fibonacci sequence occurs frequently in nature and has numerous applications in applied and pure mathematics.

The Fibonacci sequence is the sequence of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$$

where each member of the sequence is the sum of the preceding two. Therefore, the n th Fibonacci number is defined recursively as follows:

$$\begin{aligned} y_1 &= y_2 = 1 \\ y_n &= y_{n-1} + y_{n-2} \quad n \geq 3 \end{aligned} \tag{1}$$

Historically, this sequence appeared for the first time in a problem posed by the Italian scholar Leonardo Fibonacci in 1202. In his famous work *Liber Abaci*, Leonardo Fibonacci asked the following famous question on the rate growth of rabbits:

Suppose that on January, 1st there are two newborn rabbits, one male and one female. What is the number of rabbits produced in a year if the following conditions hold:

- 1) each pair takes one month to reach maturity*
- 2) each pair produces a mixed pair of rabbits every month, from February on; and*
- 3) no rabbits die during the course of the year.*

Assume that on January 1st there is one mixed pair of baby rabbits. At the beginning of February there will still be one pair of rabbits since it takes a month for them to become mature and reproduce. In February one mixed pair of rabbits will be produced and in March two pairs will be produced, one by the original pair and one by the pair produced in February. Following this pattern, in April, three pairs will be produced, and in May five pairs. More compactly,

TABLE 1
Solution of the Fibonacci Problem on Rabbits

Number	Month	Number of Adult Pairs	Number of Baby Pairs	Total
1.	January	0	1	1
2.	February	1	0	1
3.	March	1	1	2
4.	April	2	1	3
5.	May	3	2	5
6.	June	5	3	8
7.	July	8	5	13
8.	August	13	8	21
9.	September	21	13	34
10.	October	34	21	55
11.	November	55	34	89
12.	December	89	55	144
13.	January	144	89	233

Source:

Under the conditions of the problem, the total number of pairs of rabbits one year later will be 233. The entries in the last column of Table 1 are the first thirteen members of the Fibonacci sequence.

II. The Fibonacci Sequence in Matrix Form

Although the recursion definition in the introduction gives a complete description of the Fibonacci sequence, it is not particularly useful if we are interested in finding an arbitrary Fibonacci number y_k . For example, if we want to compute y_{1000} using the definition, we shall need to compute all 999 members that precede it. Therefore, we derive an explicit formula for the k th Fibonacci number as a function of k by solving the recurrence relation in matrix form.

Let us modify the original definition by adding the following trivial equation:

$$\begin{aligned} y_{k+1} &= y_{k+1} \\ y_{k+2} &= y_{k+1} + y_k \end{aligned} \tag{2}$$

This system of equations can be written in matrix form by letting,

$$u_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and noticing the relation,

$$u_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

we can rewrite the matrix equation in the general form:

$$u_{k+1} = Au_k \tag{3}$$

The vector \mathbf{u}_k can be written in terms of \mathbf{u}_0 by noting the relation,

$$\begin{aligned}\mathbf{u}_1 &= A\mathbf{u}_0 \\ \mathbf{u}_2 &= A\mathbf{u}_1 = A^2\mathbf{u}_0 \\ \mathbf{u}_3 &= A\mathbf{u}_2 = A^3\mathbf{u}_0 \\ &\cdot \\ &\cdot \\ \mathbf{u}_k &= A^k\mathbf{u}_0\end{aligned}$$

Since y_k is the top entry in \mathbf{u}_k and the matrices A and \mathbf{u}_0 are known, we can readily use this equation to find an arbitrary Fibonacci number. For example, y_5 can be computed from the equation $\mathbf{u}_5 = A^5\mathbf{u}_0$:

$$\mathbf{u}^5 = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} y_5 \\ y_6 \end{bmatrix}$$

However, this formula is useful in practice only if we can find a convenient way to compute A^k for higher values of k . Diagonalization of A provides a possible solution to this practical problem.

First, in order to determine whether A is diagonalizable, we derive and solve the characteristic equation of A :

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \quad (4)$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Next we solve for the eigenvectors,

$$(A - \lambda_i I)x_i = 0 \quad (5)$$

For λ_1 ,

$$\begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 & 0 \\ 1 & 1-\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{bmatrix} \xrightarrow{ref} \begin{bmatrix} 1 & \left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$$

For λ_2 ,

$$\begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 & 0 \\ 1 & 1-\left(\frac{1-\sqrt{5}}{2}\right) & 0 \end{bmatrix} \xrightarrow{ref} \begin{bmatrix} 1 & \left(\frac{1-\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}$$

Since the 2x2 matrix A has two distinct eigenvalues, it follows that A is diagonalizable and there exist a diagonal matrix D and an invertible matrix P, such that:

$$A = \mathbf{HP}^{-1}, \quad (6)$$

where D , P and P^{-1} are constructed as follows:

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}, \quad P = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & -\left(\frac{1+\sqrt{5}}{2}\right) \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$$

Since A is diagonalizable, A^k can be easily computed by applying the formula:

$$A^k = \mathbf{D}^k \mathbf{P}^{-1}, \quad (7)$$

In this way computations are significantly facilitated since D is a diagonal matrix and computing D^k does not pose any serious technical difficulties.

From equation (5), we can derive a general formula for A^k in terms of λ_1 and λ_2 .

$$A^k = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix},$$

Multiplying the three matrices out and simplifying yields,

$$A^k = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2^k \lambda_1 - \lambda_1^k \lambda_2 & \lambda_2^{k+1} \lambda_1 - \lambda_1^{k+1} \lambda_2 \\ \lambda_1^k - \lambda_2^k & \lambda_1^{k+1} - \lambda_2^{k+1} \end{bmatrix} \quad (8)$$

From equation (6), the k-th Fibonacci number y_k can be found from the relation,

$$u_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = A^k u_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2^k \lambda_1 - \lambda_1^k \lambda_2 & \lambda_2^{k+1} \lambda_1 - \lambda_1^{k+1} \lambda_2 \\ \lambda_1^k - \lambda_2^k & \lambda_1^{k+1} - \lambda_2^{k+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2^{k+1} \lambda_1 - \lambda_1^{k+1} \lambda_2 \\ \lambda_1^{k+1} - \lambda_2^{k+1} \end{bmatrix}$$

Hence, $y_k = \frac{1}{\sqrt{5}} (\lambda_2^{k+1} \lambda_1 - \lambda_1^{k+1} \lambda_2)$. Using the fact that the two eigenvalues have the property, $\lambda_1 \lambda_2 = -1$, the expression for y_k simplifies to:

$$y_k = \frac{1}{\sqrt{5}} (\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right] \quad (9)$$

The formula above was discovered in 1843 by the French mathematician J.P.M.Binet (1786-1856) and is known today in the literature as the Formula of Binet.

The second term of Binet's formula is less than one in absolute value, so as k gets progressively larger, it will approach zero. Therefore,

$$\lim_{k \rightarrow \infty} y_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k \right] \quad (10)$$

Hence, the sequence of ratios of consecutive Fibonacci numbers (y_{k+1} / y_k) is convergent and its limit is given by:

$$\varphi = \lim_{k \rightarrow \infty} \frac{y_{k+1}}{y_k} = \left(\frac{1 + \sqrt{5}}{2} \right) \approx 1.6180339887 \quad (11)$$

The limit φ is called the *Golden Mean*, and has been regarded since ancient times as the aesthetically ideal ratio of width to height for a rectangle. It is commonly reflected in natural objects which grow by a linear increment (e.g. snail shells, sunflowers, and in great many other places).

III. Generalizations of the Fibonacci sequence. Lucas sequence.

The Fibonacci sequence is a special type of a broader class of recurrence relations that occur frequently in applied mathematics. A k -th order linear homogenous recurrence relation is a relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad , \quad (12)$$

where $c_1, c_2, \dots, c_k \in R$.

In particular, the Fibonacci sequence is a second order sequence with $c_1=1$ and $c_2=1$.

The analysis applied for the Fibonacci sequence in section II can be readily generalized for any second order linear recurrence relation with constant coefficients of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad (13)$$

where c_1 and c_2 are non zero constants. A sequence derived from this equation is often called a Lucas sequence. It is an interesting question to explore whether the method we used for deriving the formula for y_k of the Fibonacci sequence can be applied to find a similar formula for y_k of the Lucas sequence.

Let us start with several examples:

1. Consider the following Lucas sequence:

$$y_{k+2} = 3y_{k+1} - 2y_k, \quad (14)$$

where $y_0=0$ and $y_1=1$.

The first few terms generated by the equation above are 0, 1, 3, 7, 15, 31, 63.....

In order to find an explicit formula for y_k , we can repeat the same method we used for deriving the formula of Binet for the Fibonacci sequence.

First, we add the trivial equation:

$$\begin{aligned} y_{k+1} &= y_{k+1} \\ y_{k+2} &= 3y_{k+1} - 2y_k \end{aligned} \quad (15)$$

By letting,

$$u_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

and noting,

$$A u_k = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} y_{k+1} \\ 3y_{k+1} - 2y_k \end{bmatrix} = u_{k+1} \quad (16)$$

the system of equations (15) can be written in matrix form as

$$\mathbf{u}_{k+1} = A\mathbf{u}_k \quad (17)$$

The vector \mathbf{u}_k could be written in terms of \mathbf{u}_0 by noting that

$$\mathbf{u}_k = A\mathbf{u}_{k-1} = AA\mathbf{u}_{k-2} = \dots = A^k \mathbf{u}_0 \quad (18)$$

As in the case for the Fibonacci sequence, the main goal is to find a general formula for A^k by possibly diagonalizing A .

In order to diagonalize A , we derive and solve the characteristic equation of A :

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & 3 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \quad (19)$$

$$\lambda_1 = 2 \quad \lambda_2 = 1$$

Since A has two distinct eigenvalues, it is diagonalizable and can be expressed in the form $A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D . In fact, the columns of P are the eigenvectors of A corresponding to λ_1 and λ_2 , and the non-zero entries in D are the two eigenvalues.

The eigenvectors corresponding to the two eigenvalues are computed as follows.

For $\lambda_1 = 2$,

$$\begin{bmatrix} -2 & 1 & 0 \\ -2 & 3-2 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$,

$$\begin{bmatrix} -1 & 1 & 0 \\ -2 & 3-1 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let us construct:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix},$$

Since A is diagonalizable,

$$A^k = P D^k P^{-1} = \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2-2^k & 2^k-1 \\ 2-2^{1+k} & -1+2^{1+k} \end{bmatrix}, \quad (20)$$

From equation (16),

$$u_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = A^k u_0 = \begin{bmatrix} 2-2^k & 2^k-1 \\ 2-2^{1+k} & -1+2^{1+k} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k-1 \\ 2^{1+k}-1 \end{bmatrix} \quad (21)$$

Hence,

$$y_k = 2^k - 1 \quad (22)$$

In order to verify the obtained formula, we plug in $k = 0, 1, 2, 3, 4..$ in () and get 0, 1, 3, 7, 15.., which are exactly the first few terms of the Lucas sequence defined in example 1. In this case, the method used to obtain the formula for the Fibonacci sequence works well and yields an explicit formula for the k -th term of the Lucas sequence.

However, let us consider the following example:

2. Consider the following Lucas sequence generated by the equation:

$$y_{k+2} = 2y_{k+1} - y_k, \quad (23)$$

where $y_0 = 0$ and $y_1 = 1$.

The first few terms this sequence are 0, 1, 3, 7, 15, 31, 63.....

Following the same steps as in example 1, the equation above can be also written in matrix form. Let

$$u_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

Then,

$$A u_k = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} y_{k+1} \\ 2y_{k+1} - y_k \end{bmatrix} = u_{k+1}, \quad (24)$$

Hence equation (23) can be stated compactly as

$$u_{k+1} = A u_k, \quad (25)$$

or alternatively as

$$u_k = A^k u_0 \quad (26)$$

In order to diagonalize A, we solve its characteristic equation,

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad (27)$$

$$\lambda_{1,2} = 1$$

Since the 2x2 matrix A has only one distinct eigenvalue with multiplicity 2, it follows that A is not diagonalizable. Therefore, there is no general formula for A^k and hence for y_k .

From the two examples above we can derive the following two conclusions about our matrix method for solving second order recurrence relations:

- 1) If the characteristic equation of A has two distinct solutions, then A is diagonalizable and we can derive an explicit formula for y_k .
- 2) If the characteristic equation of A has less than two distinct solutions, then A is not diagonalizable and our method for deriving y_k cannot be applied.

More formally, our conclusions are consistent with the following more general theorem:

Theorem: Let γ and δ be the distinct solutions of the equation $x^2 - \alpha x - b = 0$, where $a, b \in R$ and $b \neq 0$. Then every solution of the linear homogenous recurrence relation with constant coefficients $a_n = \alpha a_{n-1} + b a_{n-2}$, where $a_0 = C_0$ and $a_1 = C_1$, is of the form

$$a_n = A\gamma^n + B\delta^n$$

for some constants A and B .¹

IV. Conclusion

This paper discussed how linear algebra can be applied to the analysis of linear recurrence relations, a number of which arise frequently in applied and pure math. In particular, it emphasized the concepts of eigenvalues and eigenvectors and how helpful they are in describing and understanding infinite sequences of integers.

Bibliography:

¹ For the proof of this theorem see Koshy, p.143-4