

How many squares are there on a chessboard?

The aim of this investigation is to find out how many squares and rectangles of specific sizes can be found on a chessboard, and too see if there is a common sequence and algebraic formulae for each example.

I have started with the simplest example that is to count the different combinations of squares on a range of boards from 2x2 board up to an 8x8 board.

Results from the count

2x2 board

$$1 \times 1 = 4$$

$$2 \times 2 = 1$$

3x3 board

$$1 \times 1 = 9$$

$$2 \times 2 = 4$$

$$3 \times 3 = 1$$

4x4 board

$$1 \times 1 = 16$$

$$2 \times 2 = 9$$

$$3 \times 3 = 4$$

$$4 \times 4 = 1$$

5x5 board

$$1 \times 1 = 25$$

$$2 \times 2 = 16$$

$$3 \times 3 = 9$$

$$4 \times 4 = 4$$

$$5 \times 5 = 1$$

6x6 board

$$1 \times 1 = 36$$

$$2 \times 2 = 25$$

$$3 \times 3 = 16$$

$$4 \times 4 = 9$$

$$5 \times 5 = 4$$

$$6 \times 6 = 1$$

7x7 board

$$1 \times 1 = 49$$

$$2 \times 2 = 36$$

$$3 \times 3 = 25$$

$$4 \times 4 = 16$$

$$5 \times 5 = 9$$

$$6 \times 6 = 4$$

$$7 \times 7 = 1$$

8x8 board

$$1 \times 1 = 64$$

$$2 \times 2 = 49$$

$$3 \times 3 = 36$$

$$4 \times 4 = 25$$

$$5 \times 5 = 16$$

$$6 \times 6 = 9$$

$$7 \times 7 = 4$$

$$8 \times 8 = 1$$

When the 8x8 boards results are analyzed a quadratic sequence can be identified i.e the second difference is a constant.

term	No of squares	1st diff	2nd diff
1	64		
2	49	15	
3	36	13	2
4	25	11	2
5	16	9	2
6	9	7	2
7	4	5	2
8	1	3	2

Using the general term for a quadratic sequence where $A = \frac{1}{2}$ the constant 2nd difference, which is $2 \times \frac{1}{2} = 1$

$$Y_n = An^2 + Bn + C$$

$$\text{So } y_1 = 1 \times 1 + B \times 1 + C = 64$$

$$\text{So } B + C = 64 - 1$$

$$B + C = 63$$

$$Y_2 = 1 \times 2 + B \times 2 + C = 49$$

$$4 + 2B + C = 49$$

$$\text{so } 2B + C = 45$$

we now solve the simultaneous equation

$$B + C = 63$$

$$2B + C = 45$$

$$\therefore -B = 18 \text{ OR } B = -18$$

$$\text{if } B = -18 \text{ then } C = 63 - (-18) = 81$$

so the formula is

$$y_n = n - 18n + 81$$

$$\text{if we factorize the equation } y_n = (n-9)(n-9) \therefore y_n = (n-9)^2$$

To prove the formula is correct use any term in the sequence, I will use the fourth term.

$$y_4 = 4 - 18 \times 4 + 81$$

$$= 16 - 18 \times 4 + 81$$

$$= 25$$

It can be seen from these results that the fourth term on an 8x8 board is indeed 25 thus proving the formula is correct.

Using the factorized formula $y_n = (n-9)^2$ we can further prove using the fifth term

$$y_5 = (5-9)^2 = 16 \text{ it can be seen from the table the fifth term is 16.}$$

The second investigation involves counting the number of non-overlapping rectangles of a given size on a certain size chessboard.

RESULTS

2x2 board

$$1 \times 2 = 2$$

3x3 board

$$1 \times 2 = 4$$

$$1 \times 3 = 3$$

$$2 \times 3 = 1$$

4x4 board

$$1 \times 2 = 8$$

$$1 \times 3 = 5$$

$$1 \times 4 = 4$$

$$2 \times 3 = 2$$

$$2 \times 4 = 2$$

$$3 \times 4 = 1$$

5x5 board

$$1 \times 2 = 12$$

$$1 \times 3 = 7$$

$$1 \times 4 = 6$$

$$1 \times 5 = 5$$

$$2 \times 3 = 3$$

$$2 \times 4 = 2$$

$$2 \times 5 = 2$$

$$3 \times 4 = 1$$

$$3 \times 5 = 1$$

$$4 \times 5 = 1$$

6x6 board

$$1 \times 2 = 18$$

$$1 \times 3 = 12$$

$$1 \times 4 = 8$$

$$1 \times 5 = 7$$

$$1 \times 6 = 6$$

$$2 \times 3 = 6$$

$$2 \times 4 = 4$$

$$2 \times 5 = 3$$

$$2 \times 6 = 3$$

$$3 \times 4 = 2$$

$$3 \times 5 = 2$$

$$3 \times 6 = 2$$

$$4 \times 5 = 1$$

$$4 \times 6 = 1$$

$$5 \times 6 = 1$$

7x7 board

$$1 \times 2 = 24$$

$$1 \times 3 = 16$$

$$1 \times 4 = 10$$

$$1 \times 5 = 9$$

$$1 \times 6 = 8$$

$$1 \times 7 = 7$$

$$2 \times 3 = 6$$

$$2 \times 4 = 4$$

$$2 \times 5 = 4$$

$$2 \times 6 = 3$$

$$2 \times 7 = 3$$

$$3 \times 4 = 3$$

$$3 \times 5 = 2$$

$$3 \times 6 = 2$$

$$3 \times 7 = 2$$

$$4 \times 5 = 1$$

$$4 \times 6 = 1$$

$$4 \times 7 = 1$$

$$5 \times 6 = 1$$

$$5 \times 7 = 1$$

$$6 \times 7 = 1$$

8x8 board

$$1 \times 2 = 32$$

$$1 \times 3 = 20$$

$$1 \times 4 = 16$$

$$1 \times 5 = 11$$

$$1 \times 6 = 10$$

$$1 \times 7 = 9$$

$$1 \times 8 = 8$$

$$2 \times 3 = 10$$

$$2 \times 4 = 8$$

$$2 \times 5 = 5$$

$$3 \times 6 = 5$$

$$2 \times 7 = 4$$

$$2 \times 8 = 4$$

$$3 \times 4 = 4$$

$$3 \times 5 = 3$$

$$3 \times 6 = 2$$

$$3 \times 7 = 2$$

$$3 \times 8 = 2$$

$$4 \times 5 = 2$$

$$4 \times 6 = 2$$

$$4 \times 7 = 2$$

$$4 \times 8 = 2$$

$$5 \times 6 = 1$$

$$5 \times 7 = 1$$

$$5 \times 8 = 1$$

$$6 \times 7 = 1$$

$$6 \times 8 = 1$$

$$7 \times 8 = 1$$

A linear sequence is found using the 1x5 size rectangle the sequence is 5,7,9,11

To find the formula for this sequence I used

$$Y_n = dn + B$$

Where d=the common difference, n is the term to be found and B is the first term of the sequence minus the common difference, i.e $5 - 2 = 3$

$$Y_n = 2n + 3$$

Because we have started on a five by five board we substitute n for (m -4)

Where m = board size.

$$Y_m = 2(m-4) + 3$$

To prove this formula I used the 2nd term.

$$\begin{aligned} Y_m &= 2(6-4) + 3 \\ &= 7 \end{aligned}$$

it can be seen from my results that on a m = 6x6 board there are 7 1x5 rectangles.

When overlapping rectangles of a certain size are counted on different sized boards a quadratic sequence is found. I have used a 1x4 rectangle for this example.

RESULTS

4x4 board

$$1 \times 4 = 8$$

5x5 board

$$1 \times 4 = 20$$

6x6 board

$$1 \times 4 = 36$$

7x7 board

$$1 \times 4 = 56$$

8x8 board

$$1 \times 4 = 80$$

From these results a quadratic sequence can be found.

board size	no 1x4 rect	1st diff	2nd diff
4x4	8		
5x5	20	12	
6x6	36	16	4
7x7	56	20	4
8x8	80	24	4

Using the general term for a quadratic sequence

$$Y_n = An^2 + Bn + C$$

Where $A = \frac{1}{2} \times 2^{\text{nd}} \text{ difference} = \frac{1}{2} \times 4 = 2$

$$A=2$$

$$Y_n = 2n^2 + Bn + C$$

$$\text{So } y_1 = 2 \times 1^2 + B \times 1 + C = 8$$

$$= 2 + B + C = 8$$

$$= B + C = 8 - 2$$

$$= B + C = 6$$

$$\text{so } y_2 = 2 \times 2^2 + B \times 2 + C = 20$$

$$8 + 2B + C = 20$$

$$2B + C = 20 - 8$$

$$2B + C = 12$$

Subtract the simultaneous equations

$$2B + C = 12$$

$$B + C = 6$$

$$B = 6$$

$$C = 0$$

If $B = 6$ and $C = 0$ the formula is

$$Y_n = 2n^2 + 6n$$

Because we have started from 4x4 board this is the first term of the sequence and so n is substituted for $m-3$

$$Y_m = 2(m-3) + 6(m-3)$$

$$Y_m = 2(m-3) + 6m - 18$$

So for a 5x5 board

$$Y_2 = 2(5-3) + 6 \times 5 - 18$$

$$= 8 + 12$$

$$= 20$$

there are 20 1x4 rectangles on a 5x5 board so the formula is proven.

By counting the total number of squares from all the boards, ranging from a 1x1 board to a 8x8 board, and tabulating them, a cubic sequence is found, i.e. the third difference is constant.

Term	no of squares	1st diff	2nd diff	3rd diff
1	1			
2	5	4		
3	14	9	5	2
4	30	16	7	2
5	55	25	9	2
6	91	36	11	2
7	140	49	13	2
8	204	64	15	2

Using the general term for a cubic sequence, where $A = 1/6$ of the 3rd difference

$$A = 1/6 \times 2 = 1/3$$

$$Y_n = An + Bn + Cn + D$$

$$Y_1 = 1/3 \times 1 + B \times 1 + C \times 1 + D$$

$$= 1/3 + B + C + D = 1$$

$$= B + C + D = 1 - 1/3$$

$$= B + C + D = 2/3$$

$$y_2 = 1/3 \times 2 + B \times 2 + C \times 2 + D$$

$$= 1/3 \times 8 + B \times 4 + 2C + D$$

$$= 2 \frac{2}{3} + 4B + 2C + D = 5$$

$$= 4B + 2C + D = 5 - 2 \frac{2}{3}$$

$$= 4B + 2C + D = 2 \frac{1}{3}$$

$$\begin{aligned}
y^3 &= \frac{1}{3} \times 3 + B \times 3 + C \times 3 + D \\
&= \frac{1}{3} \times 27 + 9B + 3C + D = 14 \\
&= 9B + 3C + D = 14 - 9 \\
&= 9B + 3C + D = 5
\end{aligned}$$

Solve the simultaneous equation

1. $B + C + D = \frac{2}{3}$
2. $4B + 2C + D = 2 \frac{1}{3}$
3. $9B + 3C + D = 5$

$$\text{Subtract 2 and 1} = 3B + C = 1 \frac{2}{3} \quad 4$$

$$\text{Subtract 3 and 2} = 5B + C = 2 \frac{2}{3} \quad 5$$

$$\text{Subtract 4 and 5} = 2B = 1$$

$$\text{Therefore } B = \frac{1}{2}$$

Insert $B = \frac{1}{2}$ into equation $3B + C = 1 \frac{2}{3}$ to find C

$$\begin{aligned}
3 \times \frac{1}{2} + C &= 1 \frac{2}{3} \\
= 1 \frac{1}{2} + C &= 1 \frac{2}{3} \\
C &= 1 \frac{2}{3} - 1 \frac{1}{2} = \frac{1}{6}
\end{aligned}$$

$$C = \frac{1}{6}$$

To find D insert $B = \frac{1}{2}$ and $C = \frac{1}{6}$ into

$$\begin{aligned}
B + C + D &= \frac{2}{3} \\
= \frac{1}{2} + \frac{1}{6} + D &= \frac{2}{3} \\
\frac{1}{2} + \frac{1}{6} &= \frac{2}{3} \\
\text{so } D &= 0
\end{aligned}$$

The formula for this cubic sequence is

$$Y_n = \frac{1}{3}n + \frac{1}{2}n + \frac{1}{6}n$$

Solve for the first term

$$\begin{aligned}
Y_1 &= \frac{1}{3} \times 1 + \frac{1}{2} \times 1 + \frac{1}{6} \times 1 \\
&= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} \\
&= 1
\end{aligned}$$

Solve for the fifth term

$$\begin{aligned} Y_5 &= \frac{1}{3} \times 5 + \frac{1}{2} \times 5 + \frac{1}{6} \times 5 \\ &= 4\frac{2}{3} + 2\frac{1}{2} + \frac{5}{6} \\ &= 55 \end{aligned}$$

Predict the ninth term

$$\begin{aligned} Y_9 &= \frac{1}{3} \times 9 + \frac{1}{2} \times 9 + \frac{1}{6} \times 9 \\ &= 243 + 40\frac{1}{2} + 1\frac{1}{2} \\ &= 285 \end{aligned}$$

Predict the tenth term

$$\begin{aligned} Y_{10} &= \frac{1}{3} \times 10 + \frac{1}{2} \times 10 + \frac{1}{6} \times 10 \\ &= 333\frac{1}{3} + 50 + 1\frac{2}{5} \\ &= 385 \end{aligned}$$

Y_9 and y_{10} can now be put into the table of results and shown to be correct, i.e. the third difference is still constant

Term	No of squares	1st diff	2nd diff	3rd diff
7	140	49		
8	204	64	15	2
9	285	81	17	2
10	385	100	19	2