History and Social Context of Mathematics

Assignment 1

> *The history of mathematics is not the history of concrete events, but rather the history of abstractions. It is the history of inventing ideas, of discovering patterns – it is the history of connecting these ideas and patterns to one another and to concrete elements in our life. It is truly a history where one not only learns from the past but builds on the past – where the present relies on the past, and is essential to the future.*
>
> *(The Music of Reason, Theoni Pappas, 1995)*
>
> By considering one or more collections of "ideas and patterns" discuss the validity (or otherwise) of Pappas' view.

Introduction

Many junior mathematics students would be forgiven for believing that the Greek mathematician Pythagoras (c.580 B.C) invented the concept of the right-angled triangle, such is the delirium that surrounds his theorem concerning squares in year nine. It is interesting to note however that the right-angled triangle has been in use as an aid to construction and considered as a mathematical curiosity since as far back as 2000BC.

This paper will consider the development of number and geometry in the societies of the ancient world. We shall examine the part it has played in the Indian, Mesopotamian, Egyptian and Greek societies, with a little more information about the Pythagorean School and of course Pythagoras himself.

As the subsequent text refers to Pythagoras' Theorem and Pythagorean Triples, they are explained at this stage.

Pythagoras' Theorem

"In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides"

In other words, if the hypotenuse has length z and the other two sides x and y respectively, then:

$$z^2 = x^2 + y^2$$

Pythagorean Triples

These are groups of three numbers, which satisfy the equation of Pythagoras' Theorem.
For example: {3,4,5} {5,12,13} {7,24,25} {9,40,41} {11,60,61}

Analysis

Man's interest in geometrical patterns dates back to early prehistoric times. Examples have been found of plaited rushes to form rudimentary textile art. Additionally these arrangements have developed into clothing, tents and rugs. However, geometrical patterns have not been restricted solely to textiles. They have also formed a noticeable part of ancient architecture. Examples have been found on Mexican monuments and Peruvian architectural remains. This use of geometry to beautify their surroundings suggests that the ancients had the beginnings of later scientific geometry. The first being the ability to abstract, in other words being able to identify and duplicate the simple geometric shapes occurring naturally. The second, the ability to take a three-dimensional figure and represent it two dimensionally. Essentially these people were showing an understanding of the first principles of mapping. As man's interest in and understanding of geometry evolved, it became a tool for the solution of practical problems. For example, building a house on level ground or measuring out square fields. Various schemes were devised that allowed man to make these things happen

Babylonian Mathematics

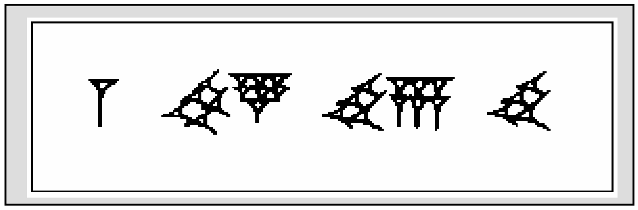
In "A History of Mathematics", Boyer emphatically states that:
"The Mesopotamian civilizations of antiquity often are referred to as Babylonian, although such a designation is not strictly correct. The city of Babylon was not at first, nor was it always in later periods, the centre of the culture associated with the two rivers [Tigris and Euphrates], *but convention has sanctioned the informal name "Babylonian" for the region during the interval from about 2000BC to roughly 600BC"*
Before discussing Babylonian geometry, we must first take a look at the history of Babylonian mathematics generally.

As far back as 3000BC, the Sumerians were building homes and temples in the area and decorating them with geometrical mosaics. They had already developed a system of cuneiform writing a millennium before and there is evidence that writing had been in existence in the area as far back as 5000BC. A significant event in the history of the area, however, comes with the invasion of Sargon the Great, the leader of the Semitic Akkadians. Under him the indigenous Sumerian and the invading Akkadian cultures were merged. Sargon's empire stretched from the Black Sea in the north to the Persian Gulf in the south. Consequently a considerable amount of information sharing subsequently took place. Thereafter followed Hittite, Asyrian and Persian invasion. All seemed to join the existing culture and the strong cultural unity remained. The use of the cuneiform script formed a strong bond between the peoples and today we have a huge collection of the baked clay tablets bearing a plethora of documents ranging from laws and school lessons to stories and personal letters.

Tablets dating from around 1700BC indicate a well-established number system. The Babylonians had taken the decimal (base10) system, used by many cultures around the world at that time, and integrated it into a sexagesimal (base 60) system. A measurement of sixty units can be divided into halves, thirds, quarters, fifths, sixths, tenths, twelfths and fifteenths more easily than a measurement of 100 units. So it appears that the sexagesimal system had been adopted to facilitate the subdividing that accompanies measuring. The efficacy of the Babylonian sexagesimal system is apparent by its survival into the twentieth century, in the form of our current measurements of angles and time.

The Babylonian numbers are shown in the attached appendix. To complement their numbers, the Babylonians developed a place value system, an extraordinary achievement, some 4000 years ago. They then saw that they could use there developed "digits" in columns representing 60^0, 60^1, 60^2, etc to represent any number.

For example the following number in cuneiform script equates to 424,000 in decimal:



60^3 = 216000	60^2 = 3600	60^1 = 60	60^0 = 1
1	57	46	40

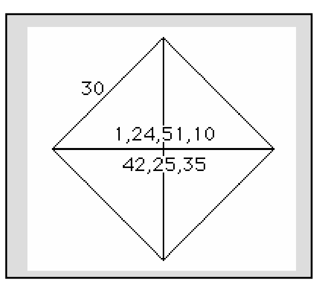
(1 x 216000)+(57 x 3600)+(46 x 60)+(40 x 1) = 424,000

The Babylonians were really quite remarkable, in their ability to understand and manage numbers. As well as their huge contribution to numeration, they developed approximation algorithms, such as the one they used for finding √2 and a process of calculating powers that echoes our modern day logarithms. A curious aspect to consider is the fact that they seemed only to investigate and develop mathematics for particular jobs in hand. There seemed to be no development of mathematics simply for pleasure, or to see if they could take something a step further. It all seemed very practical.

It is clear from some of the Sumerian tablets that the Babylonians had a good understanding of Pythagoras Theorem. The section below is an extract from a tablet held in the British Museum.

"4 is the length and 5 the diagonal. What is the breadth?
Its size is not known.
4 times 4 is 16.
5 times 5 is 25.
You take 16 from 25 and there remains 9.
What times what shall I take in order to get 9 ?
3 times 3 is 9.
3 is the breadth."

The Yale tablet, so named because of its location at Yale University shows the relationship between the sides of a square and its diagonal. A diagrammatic representation is shown below.

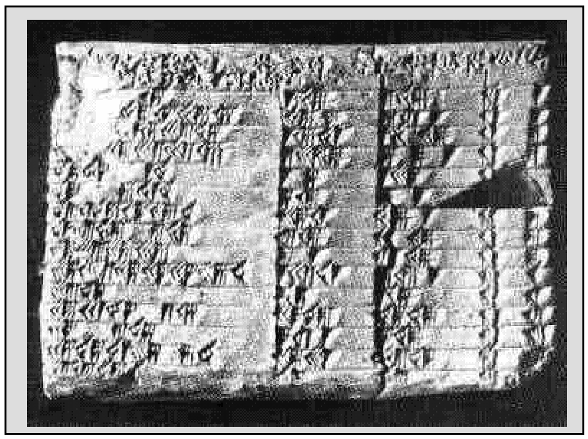


The digits represent the sexagesimal Babylonian digits.

Babylonian Values	Modern Values
$30_{60} = 30_{10}$	
$1.24,51,10_{60} = 1.4142129263_{10}$	1.414213562
$42.25,35_{60} = 42.426388889_{10}$	42.42640687

The diagonal of the square shows the ratio of the triangle's side to the hypotenuse, ($\sqrt{2}$) with the length beneath. The side of the triangle measures 30 units. Comparing the Babylonian values with today's calculated result yields a startling result. This not only shows an understanding of the principles of Pythagoras' Theorem, but also an astonishing calculation of $\sqrt{2}$.

Amongst the many other tablets to emerge from Mesopotamia has been the so-called Plimpton 322 tablet, kept in the Plimpton collection at Columbia University. Dating from between 1800 and 1650 BC the tablet appears to show a record of commercial transactions. However under closer inspection by the mathematical historians Neugebauer and Sachs it proceeded to express a whole new meaning.



Consequently, as is the way with items from antiquity, furious debate has raged.

The tablet is divided into four columns thus:
(The numbers are shown in the original sexagesimal system, the decimal notation is shown in brackets after.)

	Y	z	Row Number
1;59,0,15 (1.9834)	1,59 (119)	2,49 (169)	1
1;56,56,58,14,50,6,15 (1.9492)	56,7 (3367)	1,20,25 (4825) *	2
1;55,7,41,15,33,45 (1.9188)	1,16,41 (4601)	1,50,49 (6649)	3
1;53,10,29,32,52,16 (1.8862)	3,31,49 (12709)	5,9,1 (18541)	4
1;48,54,1,40 (1.8150)	1,5 (65)	1,37 (97)	5
1;47,6,41,40 (1.7852)	5,19 (319)	8,1 (481)	6
1;43,11,56,28,26,40 (1.7200)	38,11 (2291)	59,1 (3541)	7
1;41,33,59,3,45 (1.6928)	13,19 (799)	20,49 (1249)	8
1;38,33,36,36 (1.6427)	8,1 (481) *	12,49 (769)	9
1;35,10,2,28,27,24,26,40 (1.5861)	1,22,41 (4961)	2,16,1 (8161)	10
1;33,45 (1.5625)	45,0 (45)	1,15,0 (75)	11
1;29,21,54,2,15 (1.4894)	27,59 (1679)	48,49 (2929)	12
1;27,0,3,45 (1.4500)	2,41 (161) *	4,49 (289)	13
1;25,48,51,35,6,40 (1.4302)	29,31 (1771)	53,49 (3229)	14
1;23,13,46,40 (1.3872)	56 (56)	1,46 (106) *	15

The fourth column simple denotes the row number of the tablet.

Columns three and four however, show possible values of z and y that satisfy our original Pythagorean equation.

For example (from row 1): $169^2 - 119^2 = 120^2$.

Although there are a couple of scribal transcription errors in the original (marked with *), the middle two columns show a list of Pythagorean Triples. An achievement in itself, until we look at the first column. A little more complex we find that it is populated with values of $\left(\frac{z}{y}\right)^2$. This could be considered to correspond to the secant function, namely $\frac{1}{\cos\theta}$. However this is not the view of the majority of mathematical historians. There is still much speculation as to the application of this column, but one fact is certain. The Babylonians not only were able to understand the relationships within a right-angled triangle. They also were able to calculate and document their findings, no doubt as they used them for practical purposes.

<u>The Indian Connection</u>

In around 1500 BC the Vedic people arrived in India from the area we now know as Iran, bringing with them the religious texts known as the Veda from which they take their name. Appended to the Veda were additional texts known as the Sulbasutras which gave the Vedic the rules they needed for constructing sacrificial altars. The altars needed to be built to very precise measurements and so accurate mathematics was necessary. One of the difficulties in researching ancient Indian mathematics is that everything we know is contained within the Sulbasutras. Therefore we do not know whether they simply used mathematics for their religious requirements or used mathematics to enhance their learning.

An aspect of the Sulbasutras is the absence of any proofs of their mathematical rules. Some rules are exact, such as the method of constructing a square of equal area to a given rectangle, whereas others, particularly methods connected with circles and the use of π are erroneous. Some sections within the Sulbasutras which give rules for the construction of right angles using lengths of cord divided into Pythagorean Triples. It is disappointing, however, that all of these triples were known during the Mesopotamian times between 2000 and 600 BC and may have percolated South or have passed across to the Vedic people from Mesopotamia.

Three of the Sulbasutras were authored by Baudhaya, Apastamba and Katyayana. They would not have been simply scribes nor would they have devised the mathematics contained within them. They would instead have been men of great learning, whose interest in the mathematics would have been purely for religious purposes. One element of the volumes identifies that they knew that the square on the diagonal of a rectangle is equal to the sum of the squares on the other two sides.

From Katyayana

"The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together."

and from Baudhaya

"The rope which is stretched across the diagonal of a square produces an area double the size of the original square."

Both echo Pythagoras but it is again suggested by some that this knowledge is derived form Mesopotamia rather than independently established.

It should be pointed out, however, that there is a remarkable approximation to √2 contained within the Sulbasutras. Namely:

"Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth"

In other words : √2 = 1+1/3+1/(3x4)-1/(3x4x34) = 577/408
Evaluated, the result is 1. 414215686, which when compared to the correct result of 1.414213562 shows that the approximation is correct to 5 decimal places.

The author B Datta in "The science of the Sulba" suggests that this approximation was arrived at by an intricate process of constructing an altar twice the size of an existing one. By creating two squares, cutting up one of them and assembling it around the original square to produce a square twice the size, the new square would have a side of √2. By dividing and manipulating the remaining pieces, it is possible to very nearly complete the altar and at the same time generate the above approximation.

One should carefully assess the level of mathematics India at the time. If one is to believe Datta, it would indicate that Indian progress was much greater than had they acquired their knowledge by simply copying the Mesopotamians.

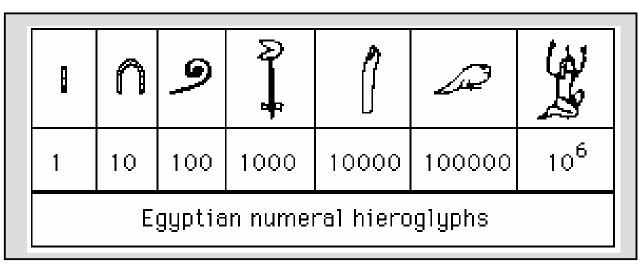
<u>Egypt</u>

It must be said that although Egyptian engineering was most impressive, Egyptian mathematics was on a much lower plane than that of the Babylonians at the same time. Most of our knowledge of Egyptian mathematics comes from two papyri. The Moscow Papyrus and the Rhind Papyrus.

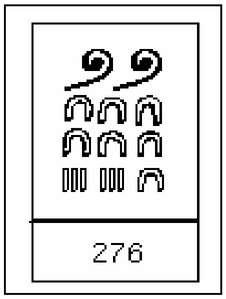
The Moscow Papyrus resides in the Museum of Fine Arts in Moscow; it was bought by V. S. Golenishchev, who died in 1947. Its author is unknown.

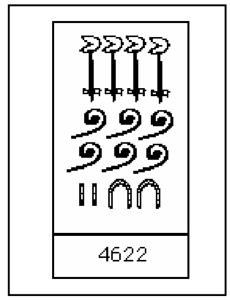
The Rhind Papyrus (also called the Ahmes Papyrus) is named after the British collector, Henry Rhind, who acquired it in 1858. It was copied by a scribe, Ahmes in around 1650 BC from another document written around 2000 BC. This possibly was in turn copied from a document from about 2650 BC, incidentally, the time of the Egyptian architect Imhotep. The man credited with the construction of Egypt's first pyramid. The Rhind Papyrus is now in the British Museum

From what we have seen of the Sumerians, they developed a logical, progressive number system based on 60. Not so with the Egyptians. Instead they evolved a base 10 system comprising hieroglyphs and symbols to represent powers of 10 and single vertical strokes to represent single units. Depending on the place value, a different symbol was used to represent the digit. Hence Egyptian numbers could appear in a variety of orientations.



Some examples of Egyptian numbers are shown below:





This indicates that although addition and subtraction may be fairly simple processes, experience of manipulating Roman Numerals will warn you that multiplication and division of Egyptian numbers was a complex business. This, however did not pose a great problem. Instead of wrestling with the difficulties of multiplication and division, the Egyptians simply adapted addition and used their numerical skill.

If, for example, they wanted to multiply two numbers together, say 35 and 17. They would form two columns thus:

✓	1	17
✓	2	34
	4	68
	8	136
	16	272
✓	32	544

The centre column simply starts with one and then doubles as we go down. The right hand column starts with 17 and again doubles as we go down. The clever part of the process is the left-hand column, which contains a tick for each component of 35,
i.e. 35 = (1+2+32). The Egyptians would then have added together the corresponding numbers in the right hand column i.e. 17+34+544 to arrive at the correct answer of 595. An admirable way of multiplying using just doubling and addition.

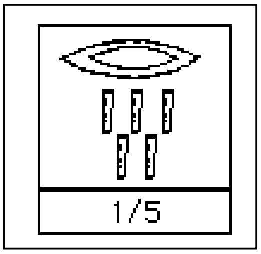
Conversely, should division be required, for example, 798/19, it would be a case of reversing the process.

	1	19
✓	2	38
	4	76
✓	8	152
	16	304
✓	32	608
	64	1216

Keep doubling 19 until you have enough to make 798 and then add up the component values in the other column, i.e. 32+8+2. The answer, of course, 42.

Not only were the Egyptians conversant with fairly hefty number crunching, but they also managed to deal with fractional quantities with proficiency. The secret of their success lies in unit fractions. Many mathematical archaeologists ask why unit fractions were used. The answer may be simply a matter of culture and convention.

It appears that when fractions were first used by the Egyptians, they restricted themselves to simple unit fractions such as ½, 1/3, ¼ and so on. The hieroglyphic representation of these was to use an oval, representing the mouth, meaning "part", atop the denominator, and omit the numerator. Essentially representing the numerator and the vinculum (dividing line) by one hieroglyph.

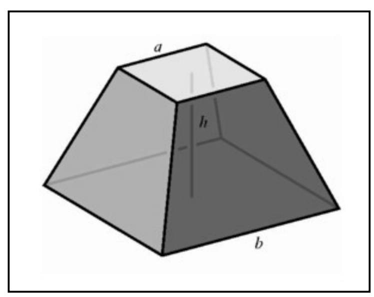


So we can see that in the years to come, when the Egyptians required a method of representing non-unitary fractions, a complete restructure of their notation would be needed. This did indeed occur with the shift from writing in hieroglyphics to writing in the more comprehensible hieratic numerals between 2200 and 1600BC.

In addition to their abilities with numeracy, the Egyptians were competent with geometry. There is still great debate as to whether their geometry is coincidental rather than intentional as sadly in comparison to wealth of tablets we have from the Babylonians we have only two papyri and a handful of other documents to substantiate any conjecture. It appears however that the Egyptians had a difficulty establishing a distinction between exact relationships and approximations. A document found at Edfu indicates a general formula for the area of any quadrilateral. To take the product of the average of opposite sides. This works fine for rectangles and squares, but when taken simply for a trapezium or any other irregular quadrilateral, it becomes completely useless. It does suggest however that the Egyptians were looking for some kind of relationship among geometric figures. It is also often also said that the Egyptians were familiar with the Pythagoras' Theorem, perhaps due to their predilection for the construction of triangular edifices. There is however, no indication of this on either the Rhind or the Moscow Papyrus. But, we should not discredit the Egyptians too much. There is a section on the Moscow papyrus which clearly shows their understanding of the calculation of the volume of a truncated pyramid. Something that should not surprise us too much.

The problem is posed:

"The base is a square of side 4, the top a square of side 2. The height is 6. Calculate the pyramid."



The solution shows the calculation:
The area of the base 4x4=16, the area of the top, 2x2=4. Then the product of the side of the base and the side of the top. 4x2=8. These three are added together. 16+4+8=28. Next, the height is divided by 3. 6/3=2. Finally, 1/3 x height x 28 to give 56.

We can see that although not formulaically expressed, the egyptians had a clear knowledge of the formula for the volume of a truncated pyramid. Namely:

$$V = \frac{1}{3}h(a^2 + ab + b^2)$$

where h is the height and a and b are the sides of the base and top respectively.

It appears in conclusion that the Egyptians had a very promising start in mathematics and geometry, but did not really progress as well as is supposed by many.

The Golden Age of the Greeks

Home of Pythagoras of Samos and Thales of Miletus, Greece is considered by many to be the location where mathematics moved from simply a tool for the solution of practical problems to the more philosophic artform it is considered by some today. It was the start of looking at problems for more esoteric purposes. Considering how a problem could be solved regardless of necessity.

Thales of Miletus was born 640BC and became a successful merchant. His travels took him to Babylon and Egypt. He may have been the cause of the cross pollination of some ideas. But it is clear that he had an interest in geometry, perhaps more from the point of view of trying to establish the reason behind facts that the Egyptians discovered empirically. He applied deductive reasoning to a variety of problems, mainly practical to satisfy his curiosity. There is no written legacy from Thales, but he will always be known as a mathematical pioneer.

In around 580BC Pythagoras was born on the Greek island of Samos. We know that he was a student of Thales, and that in about 530BC he left the island of Samos for Southern Italy. When he arrived in Italy, Pythagoras founded the Pythagorean Brotherhood, a collection of six hundred followers, who not only understood his teaching, but also contributed by adding new ideas and proofs. They lived a kind of communism supported by their patron, Milo of Croton, the wealthiest man in Croton and one of the strongest men in history – a fearsome ally. On joining the brotherhood, followers gave all their possessions to aid the group. Each member was to swear an oath of secrecy, never to reveal his or her mathematical discoveries. Indeed one member of the group was drowned after he disclosed that a new geometric solid, the dodecahedron, had been discovered. The secrecy of the brotherhood is a nuisance historically, because it clouds the evidence we do have and accounts for the overall lack of it.

One thing can be certain, Pythagoras and his brotherhood changed mathematics forever. Their aim was to study Number. They believed that Number should be treated as a god, and the closer they came to understanding it, the closer they came to the gods.

The Brotherhood did not just study number, they looked for numbers with special meaning. One of the series they discovered was the range of "perfect" numbers. Namely the range of numbers whose sum of factors equalled themselves. For example.

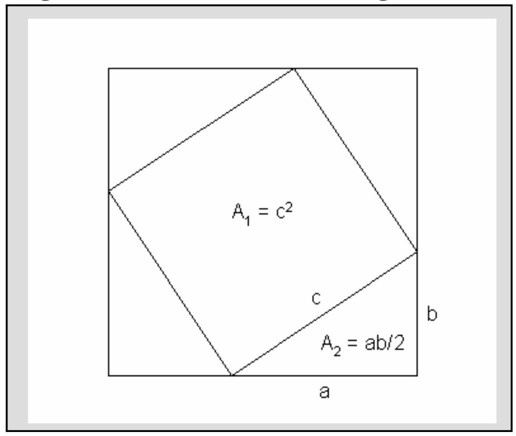
The factors of 6 are (1,2,3) - the sum of 1+2+3 equals 6.
The factors of 28 are (1,2,3,4,5,6,7) - the sum of 1+2+3+4+5+6+7 equals 28

It was not until another two hundred years later that Euclid, another greek, related this in algebraic form –

$$6=2^1 \times (2^2-1)$$
$$28=2^2 \times (2^3-1)$$
$$\ldots$$

As well as having an interest in numbers *per se*, the Pythagorean Brotherhood investigated the mathematics of musical harmonics and the planetary orbits, but perhaps the most significant work by Pythagoras is his work on the right-angled triangle. As we have already seen the theorem which bears his name was in use previously by the Babylonians, but with one important omission. The Babylonians established the rule empirically, that is to say, by discovering that for all the right-angled triangles they tried, the rule held. They did not prove that it would for work for all right angled triangles. Pythagoras, on the other hand, did.

He would simply have constructed a diagram similar to the following:



The sides of the outer square measure $(a+b) \times (a+b)$, the sides of the inner skew square measure $c \times c$.

The area of the inner square is therefore c^2 and the areas of each of the triangles is $ab/2$

Consequently,

$c^2 + 4ab/2 = (a+b)^2$

but $(a+b)^2 = a^2+b^2 +2ab$

so $c^2 +2ab = a^2 +b^2 +2ab$

which leads us to our well known equation:

$$c^2 = a^2 + b^2.$$

It is unclear exactly how Pythagoras proved the theorem, although the above solution does seem likely. Moreover, there is conjecture that it may not have been Pythagoras himself who came up with the proof, but instead another unnamed member of the brotherhood. To make matters worse, there are some, eminent Cambridge Mathematicians among them, who believe that Pythagoras himself may not even have existed!

Conclusion.

Pappas' statement that "*the history of mathematics is the history of inventing ideas*", is, I believe, borne out by the preceding dialogue. It is clear that in order to trade with adjacent communities, a satisfactory number system would have been required. It is interesting to discover that the Ancient Greeks developed a number system based on their alphabet, but due to the independence of the various island states, slight differences drifted in. (O'Connor & Robertson - St Andrews University WebPages). Ideas about number systems would have developed, as they were needed. A parallel can be drawn with the Egyptians, who in around 1800 BC developed a system of number, the hieratic numerals, which allowed them to work more efficiently.

Pappas goes on to say, "*it is the history of connecting these ideas and patterns to one another and to concrete elements in our life.*" It is clear that this naturally follows from his previous statement. Once it was seen that the ideas that had been invented had a useful purpose in simple commerce and construction, it became clear to all interconnectivity would increase efficiency. This is most evident in the Babylonians who applied their knowledge of mathematics to trade and taxation. Thereby initiating the forerunner of cash-flow forecasting.

Pappas' statement concludes that "*the present relies on the past, and is essential to the future*". I believe that this particular statement is the essence of his view. For mathematics could not possibly have developed the way it has over the last three thousand years without the determination of unnamed individuals to persevere in expanding on the developments of their forebears. It is essential that the young of today are encouraged not only to think of mathematical development in the way that it can help the current generation, but also how much of a foundation they will be building for the mathematicians of many generations to come.

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Babylonian Cuneiform Numerals