Mas3039 Mathematics: History and Culture

Topic 2: The Greek Legacy

Essay (2): Discuss Archimedes' double *reductio ad absurdum* proof for the quadrature of the parabola. Compare and contrast this to a modern calculus proof of the same result.

Archimedes of Syracuse (287 – 212 BC) is known as the greatest mathematician of his time and is considered to be one of the greatest of all time. He dominated Greek maths in the third century BC despite not being a native of the city of Alexandria, the centre of mathematical activity.[1] The son of an astronomer, Archimedes is credited with many great discoveries in mathematics, mechanics and engineering. During the second Punic war Syracuse was besieged by Romans and we are told that Archimedes invented war machines such as catapults, ropes and pulleys, and devices to set fire to the ships to keep the enemy at bay.[1] Archimedes did not think much of these inventions but it meant that mathematics and science were brought "more within the appreciation of the people in general".[2]

Archimedes' work was both productive and thoroughly detailed and he was never reluctant to share his methods of discovery. What was different about Archimedes compared to other mathematicians of his time was the fact that his work illustrated his method of discovery of a theorem prior to presenting a rigorous proof. This was to stop people claiming his work to be their own and he is quoted as saying, "those who claim to discover everything but produce no proofs of the same, may be confuted as having pretended to discover the impossible".[2]

The Method is a treatise containing a collection of Archimedes methods of discovery which was unexpectedly found in Jerusalem in the late nineteenth century.[3] The Method includes Archimedes' methods of discovery by mechanics of many important results on areas and volumes. It is the quadrature of the parabola, meaning to find the area of a segment of a parabola cut off by a chord[4], which forms the subject of the first proposition of The Method. Archimedes derives the result in two ways, firstly mechanically and secondly purely geometrically. In fact Archimedes devoted a separate treatise on the mathematical proof of this theorem.[5] The geometrical proof is based on Euxodus'

[1] C.B. Boyer, 'A History of Mathematics', Wiley (1968)
[2] Group presentation handout
[3] V.J. Katz, 'A History of Mathematics', Addison-Wesley (1998)
[4] Mactutor website: http://www-groups.dcs.st-and.ac.uk/~history/
[5] E.J. Dijksterhuis, 'Archimedes', Princeton University Press (1987)

method of exhaustion technique which means to calculate an area by approximating it by the areas of polygons. In Archimedes' proof the polygons he uses are triangles.

Archimedes cut a parabola with a chord BC creating a parabolic segment and then drew a triangle ABC whose base was the length equal to that of the chord BC. The triangle ABC leaves two segments in which Archimedes adds another two triangles ABD and ACE. Again these two triangles create four more segments in which Archimedes constructs four more triangles. Archimedes continued this process realising the more triangles he constructed, the more closely the sum of these areas were nearing the area of the parabolic segment. He also noted that the total area of the triangles created at each stage is a quarter of area of the triangles constructed in the previous stage.[3] Archimedes used this relationship to show that the area of the parabolic segment could be given by the sum of the infinite series, $X/4 + X/4^2 + X/4^3 + \ldots + X/4^n$, which is clearly $(4X)/3$, where X is the area of the initial triangle ABC.[1] However, infinite processes were frowned upon in his day[1] so Archimedes needed to prove this in another way. He completed the argument through a method called double *reductio ad absurdum*, which is Latin for "reduction to the absurd", and is also known as proof by contradiction. Archimedes assumed that $Y = (4/3)X$, X being the area of the triangle ABC, is not equal to the area of the segment, Z, so therefore Y must be greater or less than Z. Archimedes then proceeds to rule out both of these possibilities.

Firstly if Y is less than Z then triangles can be drawn in the segment, with total area T, giving $Z - T < Z - Y$. But this would imply that $T > Y$ which is impossible because the summation formula shows that $T < (4/3)X = Y$. And secondly if $Y > Z$, n is determined so that $((1/4)^n)X < Y - Z$. Since also $Y - T = (1/3)*((1/4)^n)X < ((1/4)^n)X$, it follows that $Z < T$, which is again impossible. Hence Archimedes proved by double *reductio ad absurdum,* a very common method of proof in his time, that Z cannot be more or less than $Y = (4/3)X$ meaning in fact that $Y = (4/3)X = Z$.[3]

An important lemma to this proof of Archimedes' shows how to find the sum of a geometric series and because Archimedes had no notation to express a series with arbitrarily many terms his result was given for a series of five numbers. However Archimedes' method can easily be generalised to adopt a more modern notation with n denoting an arbitrary positive integer.[3]

To summarise, Archimedes combined the method of exhaustion with a deep geometric understanding and a clever summation of a series of terms with identical successive ratios to demonstrate that the exact area of a parabolic segment is 4/3's of the area of the initial triangle inscribed into that arc.

Archimedes' work on the quadrature of the parabola was both long and detailed. It is this work by Archimedes that is considered a forerunner to modern methods of integration.[6] On this method of integration by Archimedes, Chasles said, "it gave birth to the calculus of the infinite concei ved and brought to perfection by Kepler, Cavalier, Fermat Leibniz and Newton".[2]

For all this work how much did Archimedes actually accomplish? It is true that the method of exhaustion is a work of a creative genius, but it did have two major flaws.[7] Firstly, it was not general. For each different problem, a different ingenious way of drawing triangles or some other polygon needed to be devised. Archimedes apparently was unable to find the area of a general segment of an ellipse or hyperbola.[1] The analytic approach of the modern era is completely general to the point that we do not necessarily use numbers. The second, and larger, flaw was that the method of exhaustion was not at all rigorous by modern standards. Quite simply, there was no inclusion of a limit concept. Archimedes took a segment of a parabola and filled it with some large, but finite number of polygons. The sum of the areas of these polygons would converge to the area of the figure in an easy to work with geometric series. But because of a lack of a concept of infinity Archimedes did not consider this as a series, meaning it would have been impossible for him to make the method of exhaustion at all rigorous.

It is because of these flaws in the method of exhaustion, the Greeks refusal to accept the concept of infinity, that it is easy to ignore Archimedes' work. But he did come extremely close to discovering the integral. This method of exhaustion used by Archimedes is very similar to the modern method of approximating areas of curves with simple shapes such as rectangles and trapezoids. After Archimedes it would be over 2000 years before a suitable and rigorous method of integration that was devised.

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[6] http://www.math.ubc.ca/~cass/courses/m308-02b/projects/sheppet/308Project.html
[7] http://www.math.rutgers.edu/

It wasn't until the early 17^{th} century that further developments in calculus were achieved and a modern and more rigorous calculus proof of finding the area under a curve was born from 'first principles'. A method is outlined and illustrated below.

Consider a single, very narrow strip with height y and of width δx and having area δA, where A is the total area under the curve between x coordinates a and b. For each such strip, $\delta A = y\delta x$. Taking the sum of all the areas from $x = a$ to $x = b$ we obtain the total area of $A = \sum \delta A = \sum y\delta x$

As we make δx smaller (thinner strips will yield a more accurate result for the actual area) we can write;

$A = \lim \sum y\delta x$

If $\delta A = y\delta x$, then by rearranging, $y = \delta A/\delta x$. As we δx decreases in size this relationship will also become a more accurate statement. So we can write;

$\lim \delta A/\delta x = y$.

But $\lim \delta A/\delta x$ is dA/dx

Therefore $dA/dx = y$

So y is the result of differentiating A with respect to x. Hence, to find y, we have to integrate (or work backwards) to discover what A was before differentiation. So A is the integral of y, and this is written as;

$A = \int y dx$

Where a and b represent the bounding values for x also known as the limits. The process of integration has been derived as the limiting case as the value of the x-increment, δx, in the summation tends to zero, or;

$\lim \sum y\delta x = \int y dx$

This modern calculus proof is clearly more viable as the result can be applied to finding the area under any curve. If you are given the equation of y, all you need to do is simply integrate it and evaluate it between its limits a and b. Whereas Archimedes proof by method of exhaustion is only in reference to a parabola and the result cannot be applied to ellipses or hyperbolas. The integrals arising in the quadrature of a segment of an ellipse or hyperbola require transcendental functions.[1] However the result of Archimedes proof is also very easy to use. The area under a parabola would be (4/3)*(1/2) times the base and the height of the inscribed triangle. The modern calculus proof introduces the important concepts of limits and infinity unavailable to Archimedes in his time. He was also without a number system of base 10, including the number zero, as this was introduced by The Arabians in 600AD. This would have hindered his attempts to find a more general and rigorous proof significantly.

A simple modern example using the method of exhaustion, similar to what Archimedes used, of finding the area of a parabolic segment is achieved by using rectangles rather than triangles. For example looking at the parabola $y = x^2$ below;

To find the area A_n of the parabolic segment we inscribe equally spaced rectangles in the region between 0 and x known as the limits. There will be n amount of rectangles each of width x/n. The corresponding points will be denoted by x_i, where $x_0=0$, $x_1=x/n$, $x_2=2x/n,\ldots,x_i=ix/n$. The height of the rectangles will be x_i^2 which implies that the area of each rectangle will be $(x_i^2)*(x/n)$ giving, because $x_i=ix/n$, $(x^3i^2)/n^3$ as the area of each rectangle. The last rectangle is at $x=(n-1)/n$. Now if we sum the area of the rectangles, we obtain;

$$A_n = 0^2x^3/n^3 + 1^2x^3/n^3 + 2^2x^3/n^3 +\ldots+ (n-1)^2x^3/n^3 = x^3(1/n^3)*(1^2 + 2^2 + 3^2 +\ldots+ (n-1)^2)$$

$$= x^3(1/n^3)*\sum i^2$$

The sum of the squares being;

$$1^2 + 2^2 + 3^2 +\ldots+ n^2 = n^3/3 + n^2/2 + n/6$$

This gives;

$$\sum i^2 = k(k+1)(2k+1)$$

In our case the sum ends at $k = (n-1)$, so this gives the area under a parabola to be;

$$A_n = x^3(1/n^3)*\sum i^2 = x^3(1/3 - 1/2n + 1/6n^2)$$

This holds for any x, no matter how large. Of course to get the best approximation for the area of a parabolic segment it's obvious we need to have n as being very large, that is to say we need to use as many rectangles as possible, as this reduces the error in our approximation. As n does become very large we find that the latter two terms tend to zero and the rectangles provide a better and better approximation for the area and hence $A_n = x^3/3$. This is a very interesting result because $A(x) = x^3/3$ measures the area under the graph of the function $y = x^2$ and we see that the differential of $A(x)$ gives $y = x^2$.

In summary, to find the area under the parabola $y = f(x)$ between $x = a$ and $x = b$ we divide the interval into n pieces. These pieces each have width $\Delta x = (b-a)/n$ and we call the left endpoint of these subintervals $x_i = a + i\Delta x$. Now we form a rectangle whose height is $f(x_i)$ and whose area is $f(x_i)\Delta x$. If we sum all the areas of these rectangles, we have;

$A_n = \sum f(x_i)\Delta x$

Finally, as we consider more and more rectangles, the quantity A_n gives a better and better approximation so that we may write the area as;

$A = \lim \sum f(x_i)\Delta x$

This example shares many of the same ideas introduced by Archimedes. It uses the method of exhaustion with rectangles rather than triangles, the more of these polygons the better, and a summation of a series of terms. However it also uses the concept of summation to infinity which Archimedes was unable to use.

There are many differences between Archimedes double *reductio ad absurdum* proof for the quadrature of the parabola and the modern calculus proof of the same result that I have outlined above. The modern calculus proof is effortless to understand and can easily be applied to finding areas of most segments under curves. Archimedes' proof is designed, with the formula being easy to use, for a specific case. However the modern calculus proof of finding the area under a curve was made easier to write because of the fact that mathematicians in the 17th century had a concept of what calculus was. Archimedes on the hand other may not have even realised that he was close to discovering calculus.

Archimedes' method of exhaustion may have had flaws, it was not general or rigorous, but there is no doubt that his work made a very important contribution to the evolution of calculus making him possibly the greatest mathematician of antiquity.[8]

[8] http://media.pearsoncmg.com/aw/aw_thomas_calculus_11/topics/calculus.htm

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