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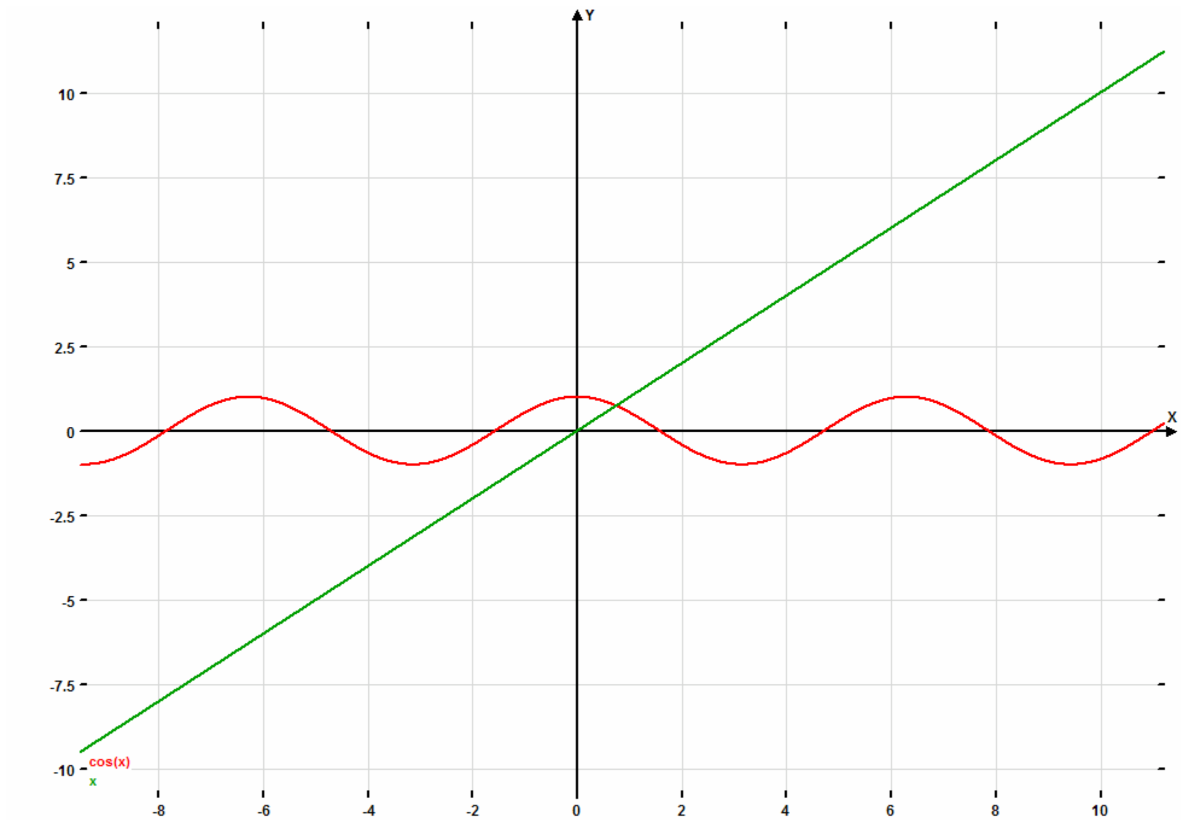
Mathematics 401 → B
18th November 2004
Internal Assessment Portfolio

Finding Zeros of Functions

Part 1

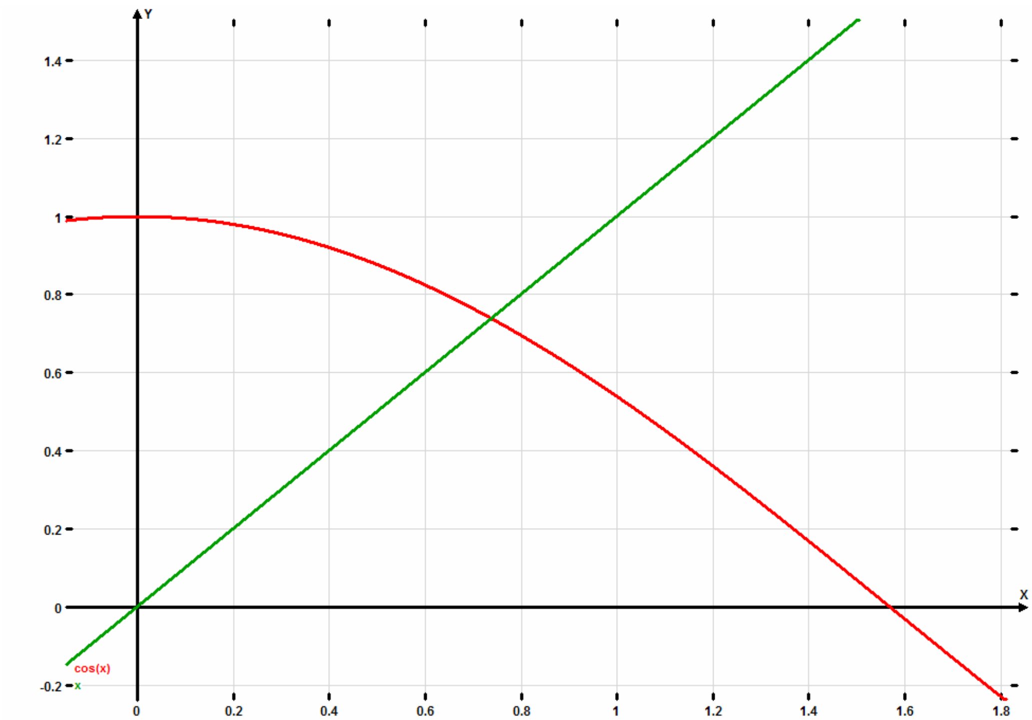
a) *Let g(x) = cos x. Show graphically that g(x) has one fixed point between x = 0 and x = 2.*



KEY

Red line	Green line
→ g(x) = cos x	→ y = x

In the introduction of the exercise, it mentions that the fixed point of the function g(x) is the point where the y = x line intersects the y = g(x) curve. And since it was given that g(x) = cos x, a substitution can be done to say that the fixed point, is the point where the y = x line intersects the g(x) = cos x curve. When a close up is made, as shown below, it can be seen that the fixed point is between x = 0 and x = 2 as suggested.



Therefore, the fixed point is (0.739, 0.739).

b) Using the iteration $x_{n+1} = g(x_n)$ and the initial value $x_1 = 1$, find the first five iterations and explain graphically why the iteration is approaching the fixed point of the function g(x).

Here, several substitutions must be made in order to graph the iteration. First, we know that when n = 0, $x_1 = 1$. When we increase the value of n by 1, $x_2 = g(x_1)$, but from part (a), we also know that g(x) = cos x. So from this, we can also state that that is the same thing as $g(x_1) = \cos x_1$. Hence, if we make the substitution, it would result to $x_2 = \cos x_1$, where we can further substitute the x_1 to 1, making the whole equation $x_2 = \cos 1$. This would then be the first iteration. Subsequently, this entire substitution process would be repeated another four times, each time using the previous answer as a substitute. The calculations below will provide a better understanding of the concept.

<u>1st iteration, n = 1</u>

$x_{1+1} = \underbrace{g(x_1)}$

$x_2 = \cos \underbrace{x_1}$

$x_2 = \cos 1 \approx 0.540$

<u>2nd iteration, n = 2</u>

$x_{2+1} = \underbrace{g(x_2)}$

$x_3 = \cos \underbrace{x_2}$

$x_3 = \cos 0.540 \approx 0.856$

<u>3rd iteration, n = 3</u>

$x_{3+1} = \underbrace{g(x_3)}$

$x_4 = \cos \underbrace{x_3}$

$x_4 = \cos 0.856 \approx 0.655$

<u>4th iteration, n = 4</u>

$x_{4+1} = \underbrace{g(x_4)}$

$x_5 = \cos x_4$

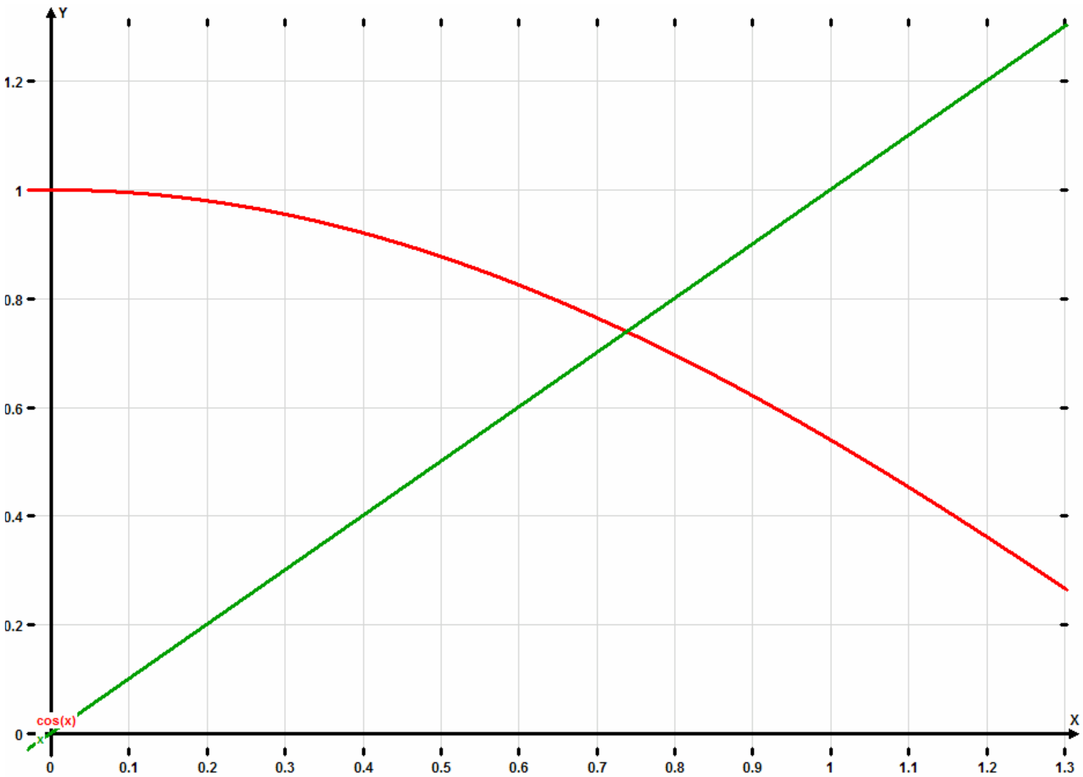
<u>5th iteration, n = 5</u>

$x_{5+1} = \underbrace{g(x_5)}$

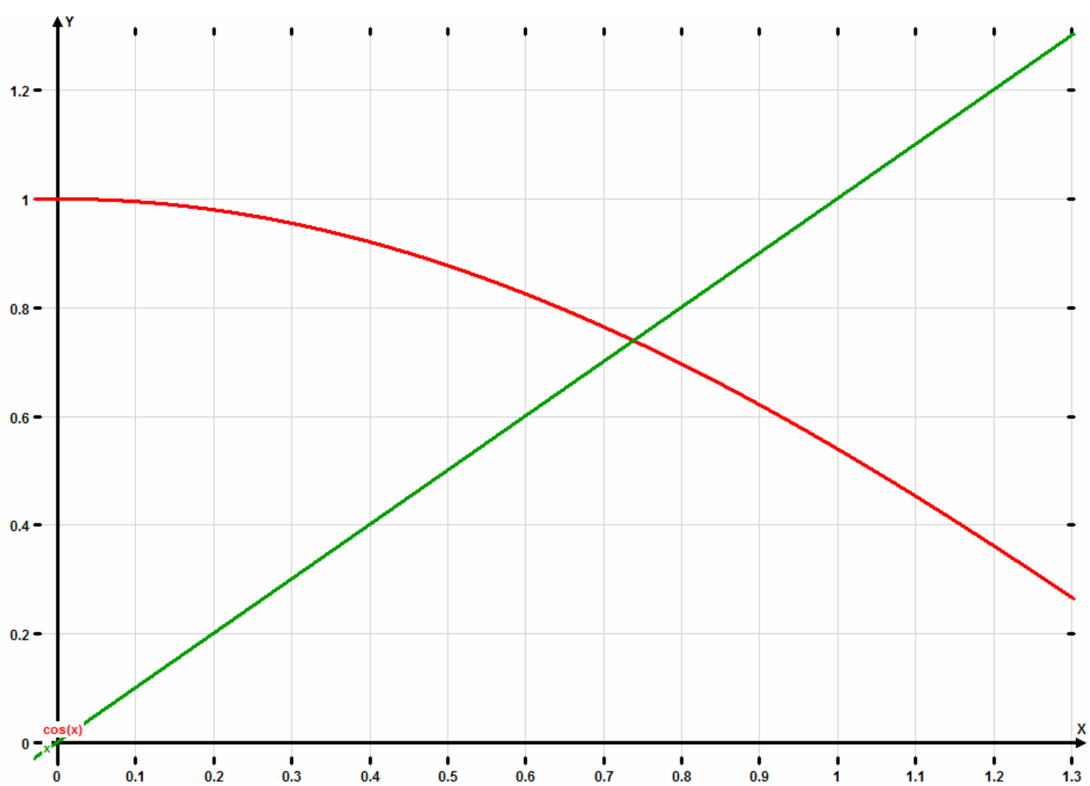
$x_6 = \cos \underbrace{x_5}$

$x_5 = \cos 0.655 \approx 0.793$ $x_6 = \cos 0.793 \approx 0.702$

From these calculations, we can see that each x value is used to find the following x value by being multiplied by cosine. When shown graphically, we start using the given $x_1 = 1$.



From that point on the x-axis, travel vertically up until the line touches the g(x) = cos x line. Then, travel horizontally, and where it crosses the y-axis is where $g(x_2)$ is. From there, go horizontally to the left, and it will cross the y = x line. When a straight line is drawn directly from that point to the x-axis, the new point on the x-axis will be the value of x2. This can then be repeated continuously until it touches the fixed point. When the graph is drawn without including the horizontal lines back to the y-axis, a diagram known as the cobweb diagram is formed, where a spiral has points which get closer to the fixed point. From the graphs, I discovered that the x values get closer and closer to the fixed point, or converge.



c) Repeat the process, but this time let g(x) = 2cosx. Comment on your results and interpret them graphically.

Here, the calculations are still the same, and the fact that $x_1 = 1$ still applies. Therefore, we do the same steps and get the results as shown below.

<u>1st iteration, n = 1</u>

$x_{1+1} = g(x_1)$

$x_2 = 2\cos x_1$

$x_2 = 2\cos 1 \approx 1.081$

<u>2nd iteration, n = 2</u>

$x_{2+1} = g(x_2)$

$x_3 = 2\cos x_2$

$x_3 = 2\cos 1.081 \approx 0.941$

<u>3rd iteration, n = 3</u>

$x_{3+1} = g(x_3)$

$x_4 = 2\cos x_3$

$x_4 = 2\cos 0.941 \approx 1.178$

<u>4th iteration, n = 4</u>

$x_{4+1} = g(x_4)$

$x_5 = 2\cos x_4$

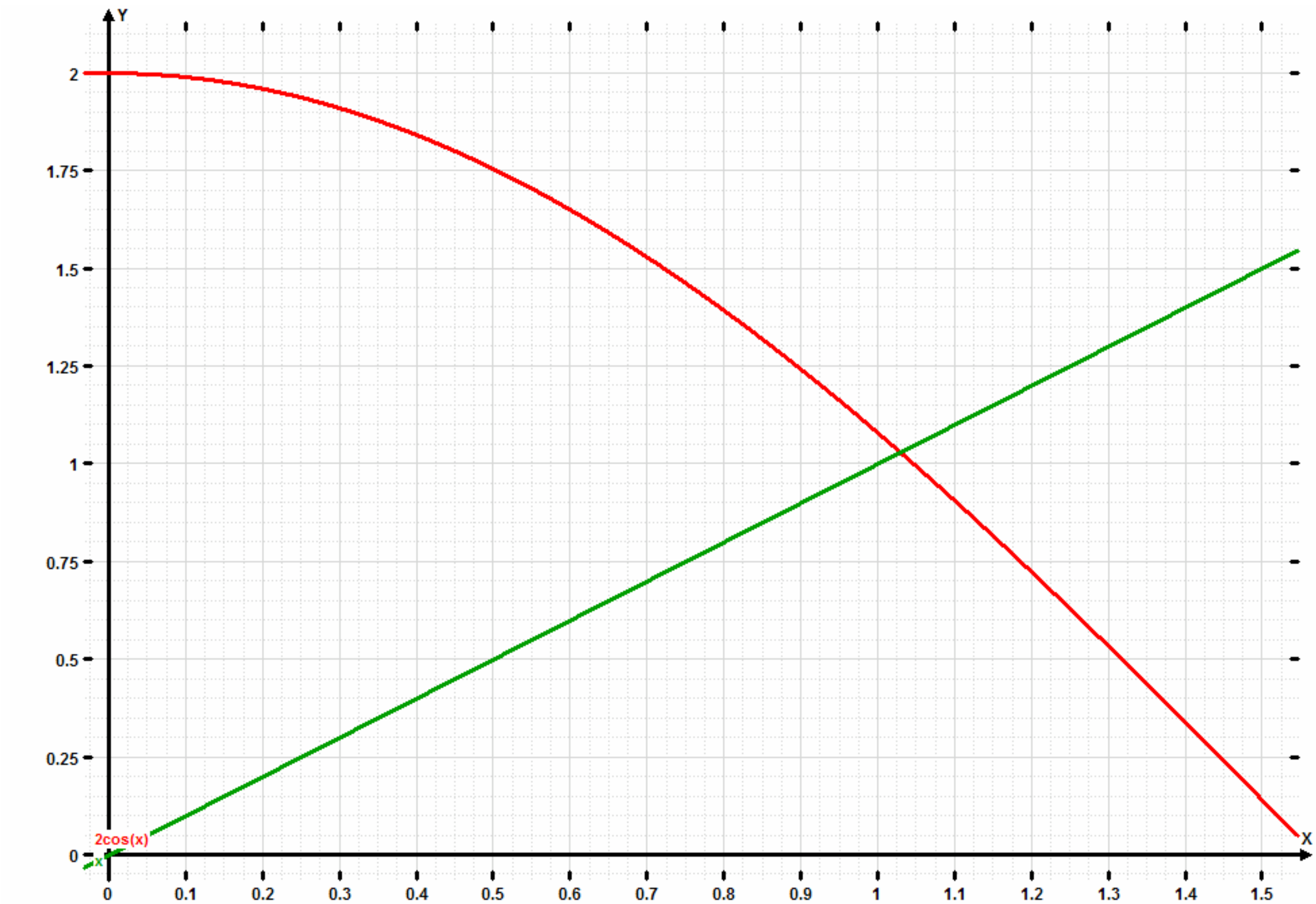
$x_5 = 2\cos 1.178 \approx 0.766$

<u>5th iteration, n = 5</u>

$x_{5+1} = g(x_5)$

$x_6 = 2\cos x_5$

$x_6 = 2\cos 0.766 \approx 1.441$



When this is graphed, it can be seen that instead of converging closer into the fixed point, it diverges out. This is interesting because it is caused just by the coefficient added to the cosine, and also because it shows that the coefficient, apart from changing where cosine crosses the y-axis, also has a different role to play. At this point, only the hypothesis that the steepness/coefficient of the cosine line is what causes the difference between whether the function converges to or diverges against the fixed point.

d) *Repeat this process for g(x) =* a *cos x, using some other values between* a *= 1 and* a *= 2 to discover which converge to a positive fixed point for g(x). Suggest why this only happens for some values of* a*.*

For this, the value of x_1 was still assumed to equal 1, while the value of a that was chosen to begin with was $a = 1.1$. Then, it was decided that the value of a would keep on increasing until 0.1 until the point where a divergence happened.

Value of *a*	**Value of x_2**	**Value of x_3**	**Value of x_4**	**Value of x_5**	**Value of x_6**	**Convergence or Divergence**
1.1	0.594	0.912	0.673	0.860	0.718	Convergence
1.2	0.648	0.957	0.691	0.925	0.722	Convergence
1.3	0.702	0.993	0.710	0.986	0.718	Convergence
1.4	0.756	1.019	0.734	1.040	0.709	Divergence

Half way through doing this, I was able to deduce that the smaller the coefficient, the further the points are to the fixed point. However, when $a = 1.4$, it was realized that somewhere in between that point and $a = 1.3$ was the point where it stopped converging and started to diverge. Thus, I decided to focus in between these two points.

Value of *a*	**Value of x_2**	**Value of x_3**	**Value of x_4**	**Value of x_5**	**Value of x_6**	**Convergence or Divergence**
1.31	0.708	0.995	0.713	0.991	0.718	Convergence
1.32	0.713	0.998	0.715	0.997	0.717	Convergence
1.33	0.719	1.001	0.717	1.003	0.715	Divergence

From this table, it is evident that the numbers that are lesser than 1.33 and some in between 1.32 and 1.33 converge to a fixed point for g(x). This happens probably due to the steepness of the cosine line.

e) *Determine for which value of* a *the iteration will converge for g(x) =* a *sin x starting with* a*=1.2. Comment on differences in the graphical representation of this iteration compared to those earlier.*

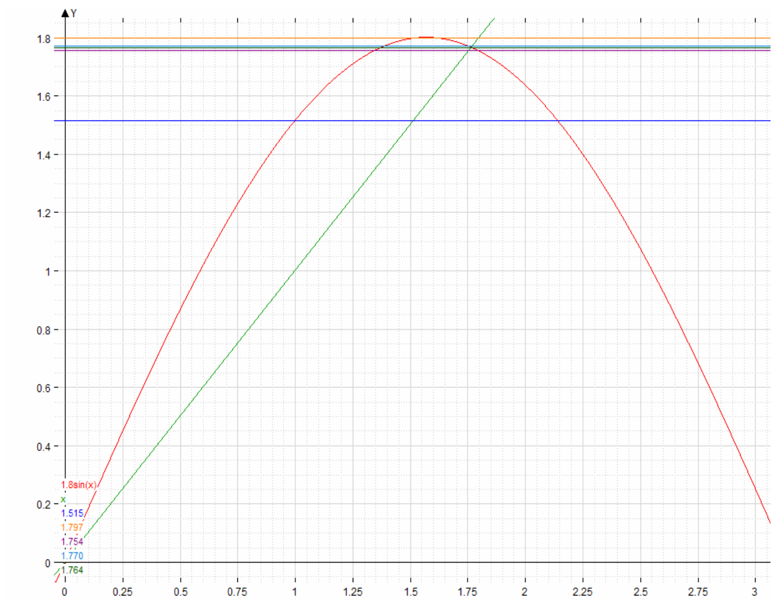
Here, we do the exact same thing as before, except that instead of using cosine, we use sine.

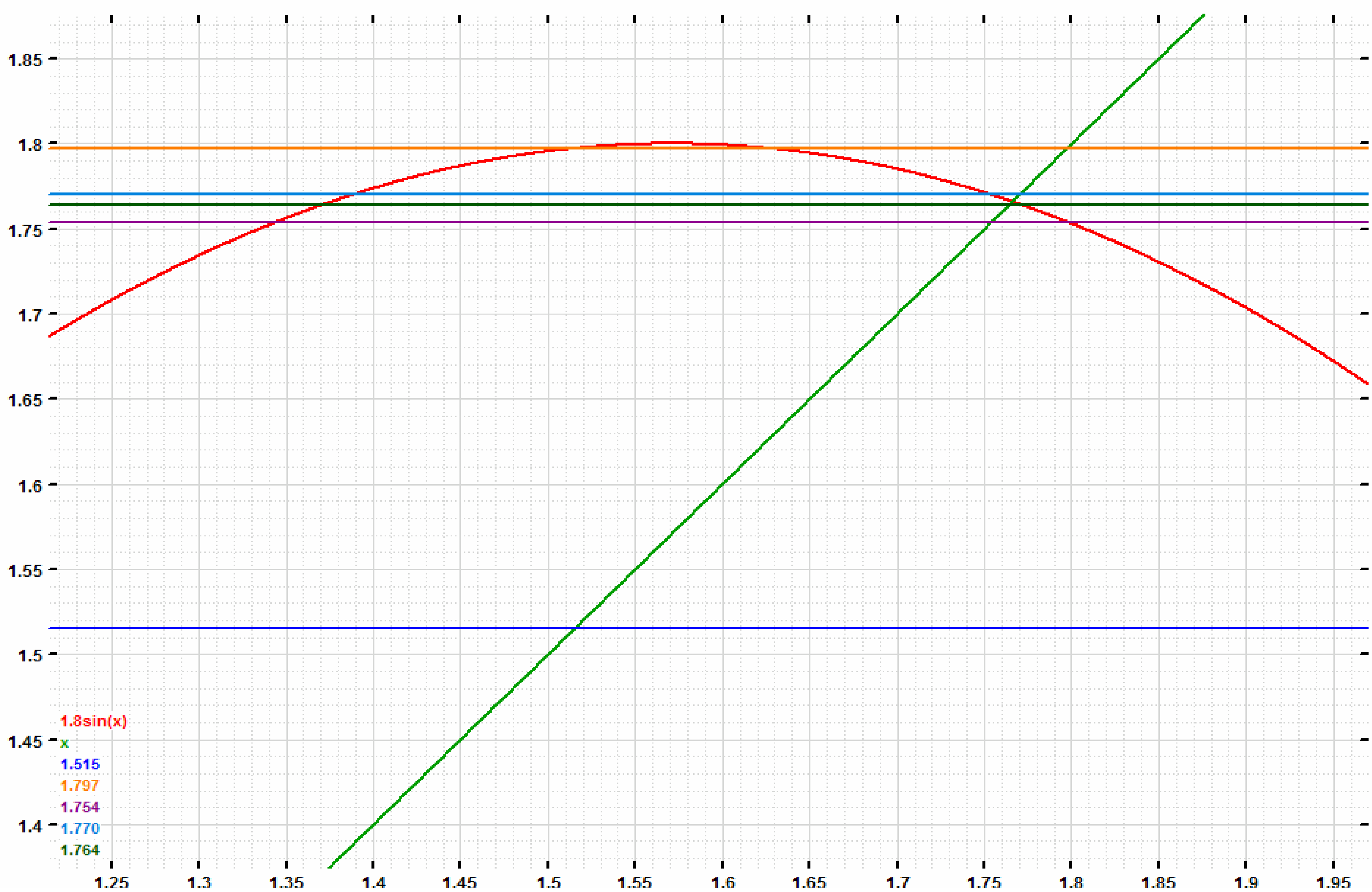
Value of *a*	**Value of x_2**	**Value of x_3**	**Value of x_4**	**Value of x_5**	**Value of x_6**	**Convergence or**

						Divergence
1.2	1.010	1.016	1.020	1.023	1.024	Convergence
1.4	1.178	1.293	1.346	1.364	1.370	Convergence
1.6	1.346	1.560	1.600	1.600	1.600	Convergence
1.8	1.515	1.797	1.754	1.770	1.764	Convergence
2	1.683	1.987	1.829	1.934	1.870	Convergence
2.2	1.851	2.114	1.883	2.094	1.906	Convergence

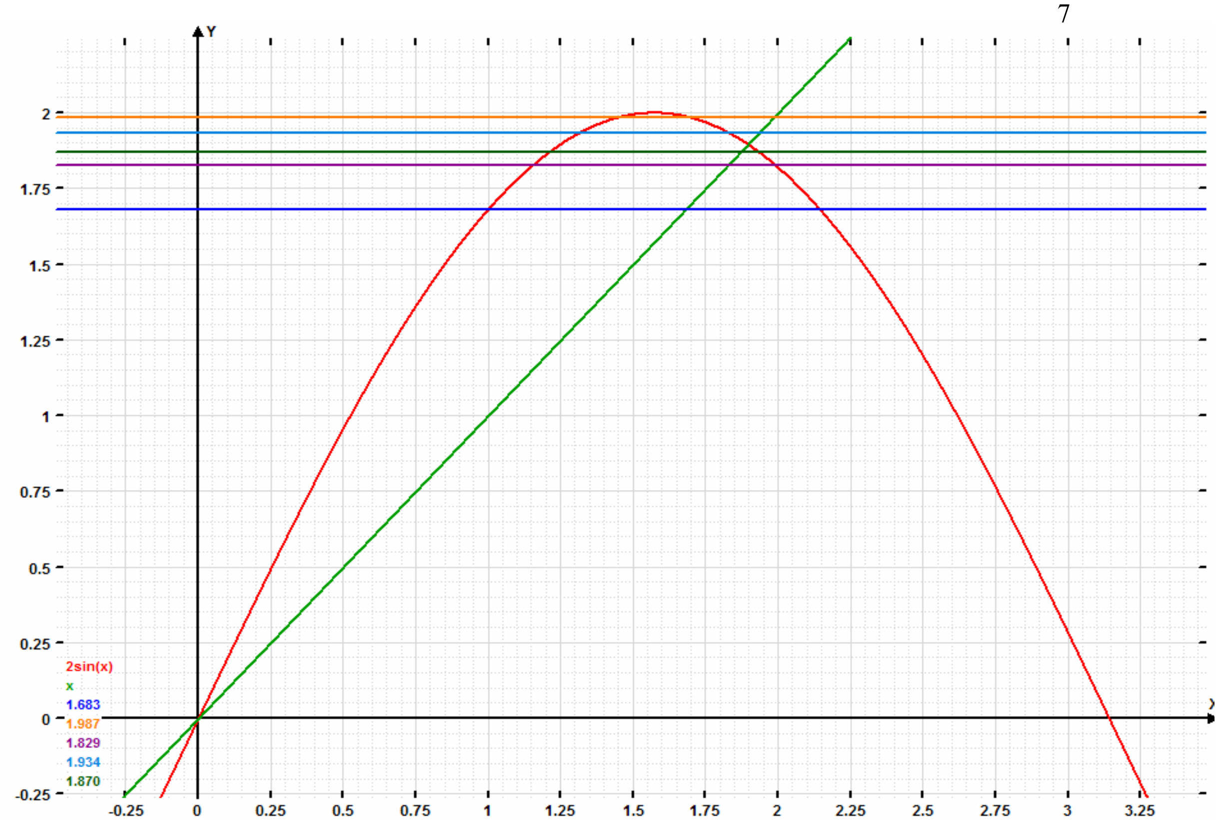
From a = 1.2 to a = 2.2, the points that are formed all converge towards the fixed point. For some of them, mainly those from 1.2 - 1.6, the x values are extremely close to each other and therefore their graphs would not show their differences and convergences clearly. However, starting from a = 1.8, it is interesting to see how the points formed actually do converge, despite the drop in the middle. This is interesting, and is due to the curve of the sine line. They are mainly different from the cosine iterations in graphical representation in that they do not form a spiral shape. The diagrams below represent the graphs when a = 1.8, a = 2, and a = 2.2.

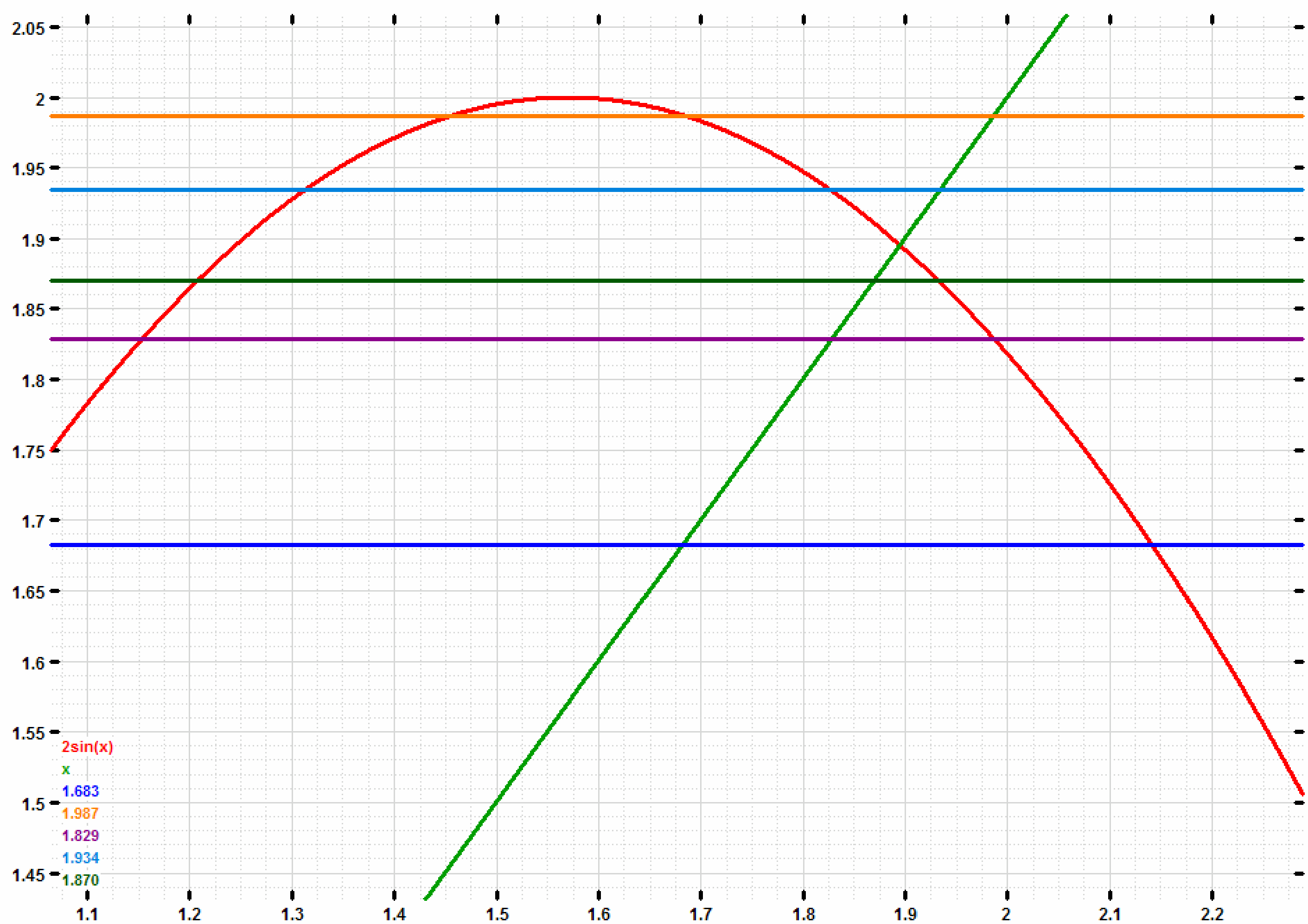
***a* = 1.8 (on the right, and bottom)**



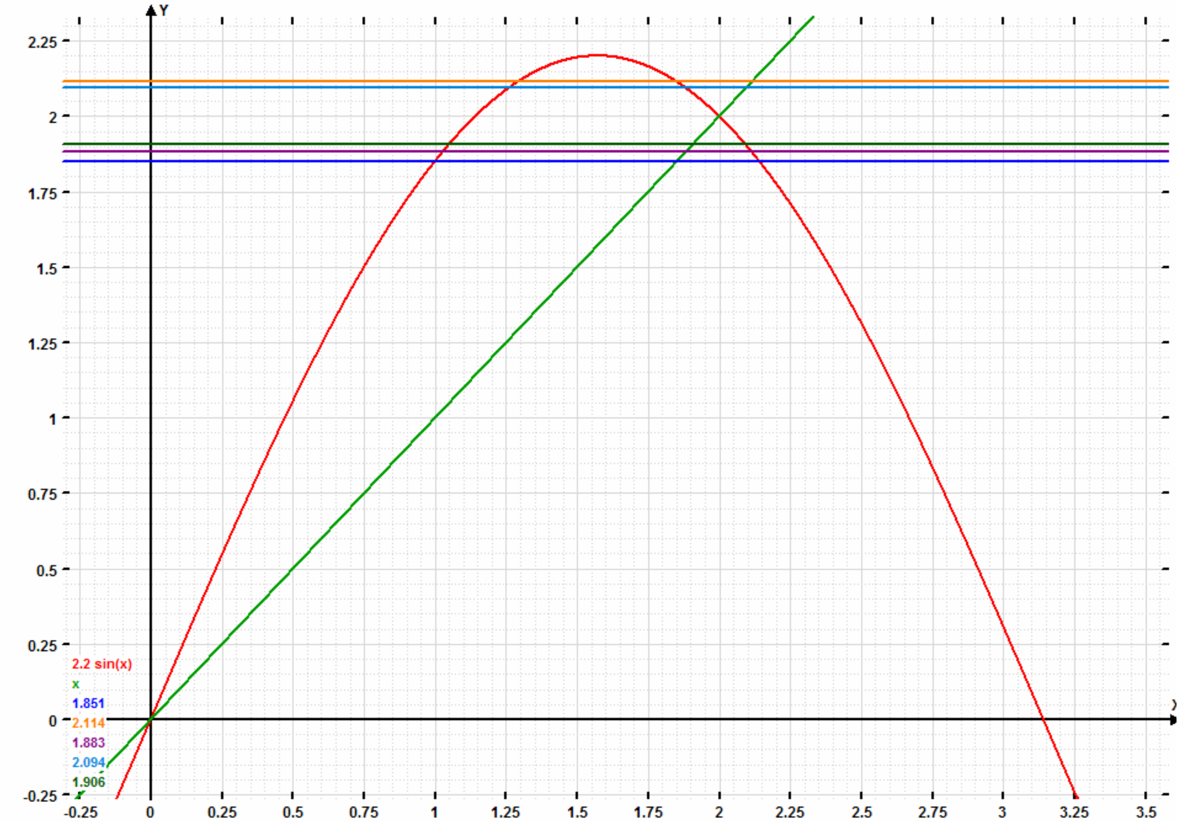


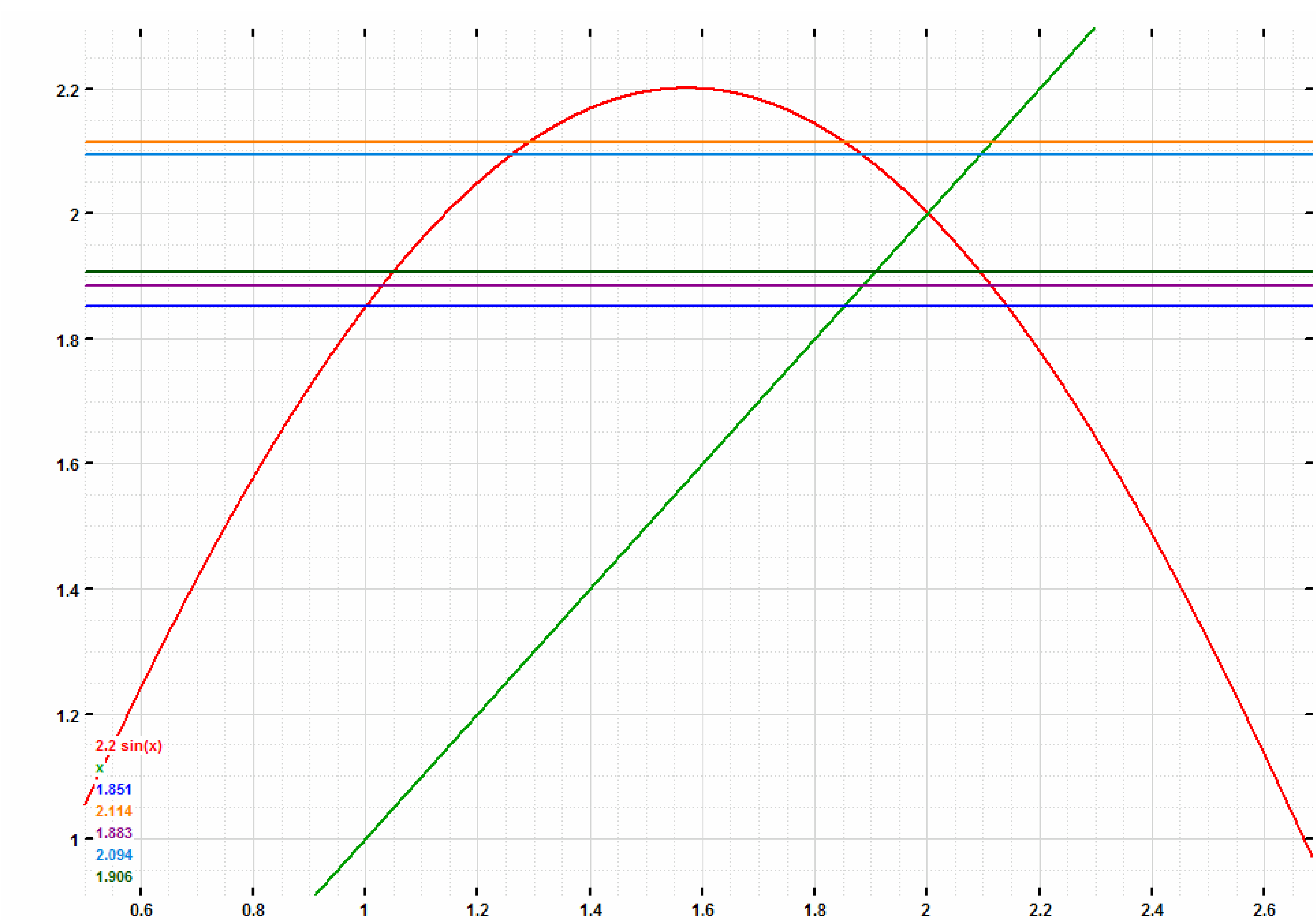
***a* = 2**


Y
2
1.75
1.5
1.25
1
0.75
0.5
0.25
0
-0.25
-0.25
0
0.25
0.5
0.75
1
1.25
1.5
1.75
2
2.25
2.5
2.75
3
3.25
2sin(x)
x
1.683
1.987
1.829
1.934
1.870


2.05
2
1.95
1.9
1.85
1.8
1.75
1.7
1.65
1.6
1.55
1.5
1.45
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
2.1
2.2
2sin(x)
x
1.683
1.987
1.829
1.934
1.870

***a* = 2.2**


Y
2.25
2
1.75
1.5
1.25
1
0.75
0.5
0.25
0
-0.25
-0.25 0 0.25 0.5 0.75 1 1.25 1.5 1.75 2 2.25 2.5 2.75 3 3.25 3.5
2.2 sin(x)
x
1.851
2.114
1.883
2.094
1.906


2.2
2
1.8
1.6
1.4
1.2
1
0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6
2.2 sin(x)
x
1.851
2.114
1.883
2.094
1.906

Part 2

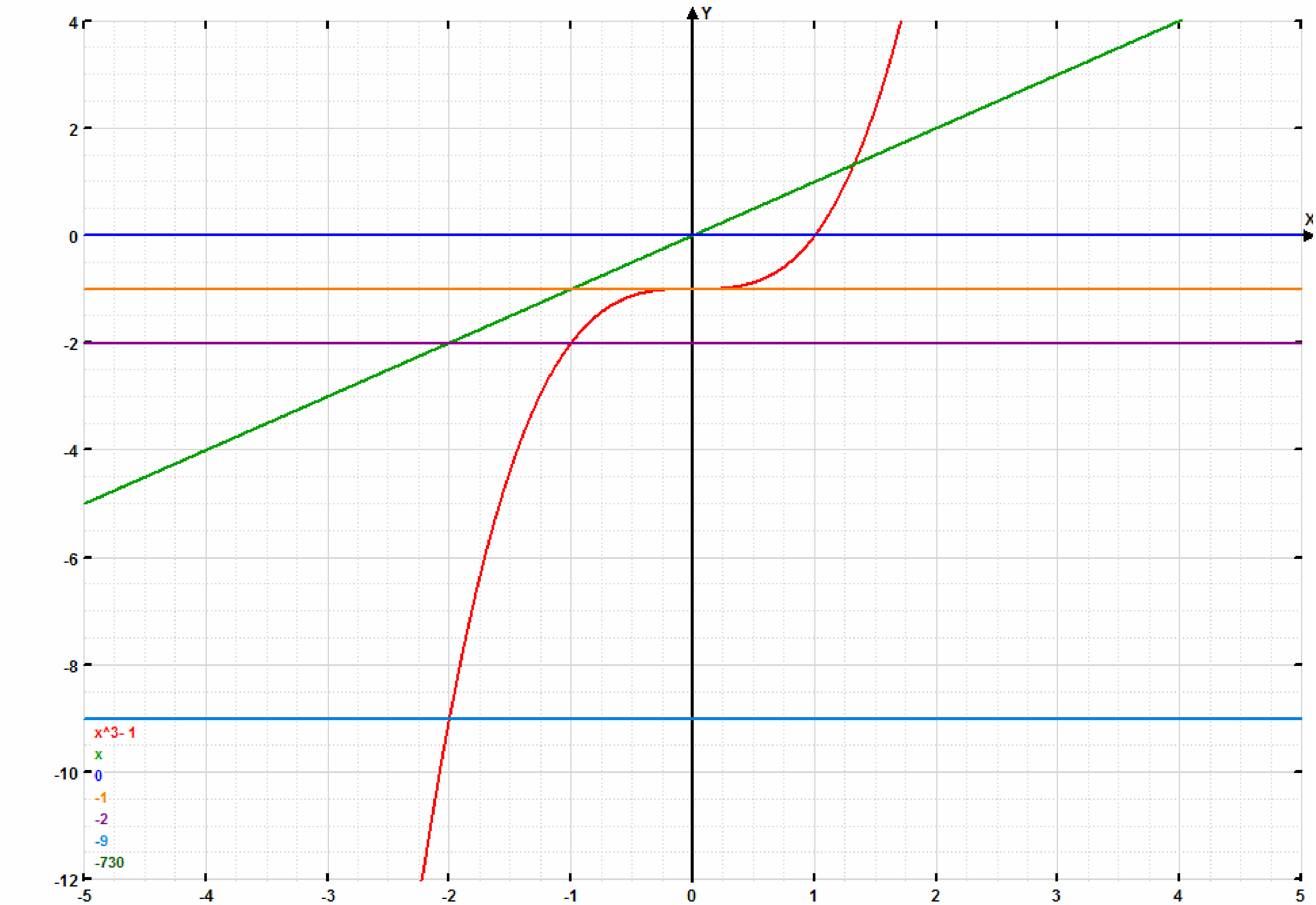
a) *Show that the equation f(x) = 0 can be rearranged in the form $x = x^3 - 1$ and use an iteration to try to find the root of the equation. Describe and explain what happened.*

The original equation that was given was $x^3 - x - 1 = 0$. To make it in the form that is stated in the question, all that needs to be done is just to add x to both sides of the equation. So now, $g(x) = x = x^3 - 1$. This would then mean that $g(x_n) = x_{n+1}$, and that $x_{n+1} = x_n^3 - 1$. Therefore, that would be the iteration. And if we stick with $x_1 = 1$, the calculations would be as followed.

$n = 1 \rightarrow x_2 = x_1^3 - 1 \rightarrow x_2 = 1^3 - 1 = 0$ $n = 2 \rightarrow x_3 = x_2^3 - 1 \rightarrow x_3 = 0^3 - 1 = -1$

$n = 3 \rightarrow x_4 = x_3^3 - 1 \rightarrow x_4 = -1^3 - 1 = -2$ $n = 4 \rightarrow x_5 = x_4^3 - 1 \rightarrow x_5 = -2^3 - 1 = -9$

$n = 5 \rightarrow x_6 = x_5^3 - 1 \rightarrow x_6 = -9^3 - 1 = -730$



In order to see if any interesting patterns had happened, I started from the point (1, 0), and I noticed that that was also the point where the function and x = 0 intersect. Then, I traced that back to the y = x line. At their intersection, I traveled directly down, and that led me to where the function and x = -1 intersected. I then traced that back to the y = x line again and traveled directly down. This also led me to where the function and x = -2 intersected. This was continued with the fourth root, and I realized that the roots diverged from the fixed point, but at the same time, they formed a very visible ladder-shape pattern. The x = -730 line is not visible in this graph because was too far apart, but the ladder-shape pattern would have continued even with that being far away from x = -9.

b) Rearrange the equation in different ways in the form x = g(x) and try new iterations, to discover which converge on the root. Explain graphically why they converge.

In order to rearrange the equation in different ways, the first thing that could be done is to add one to both sides, making the equation $x^3 - x = 1$. Then, the x on the left side of the equation can be factored out, making it, $x(x^2 - 1) = 1$, and so that only x is on one side of the equation, we divide 1 by the quantity $(x^2 - 1)$, giving us $x = 1/(x^2 - 1)$. Thus from here, we repeat how g(xn) = $x_{n+1} = 1/(x_n^2 - 1)$, however, this time we will be changing $x_1 = 2$. The calculations are as follows.

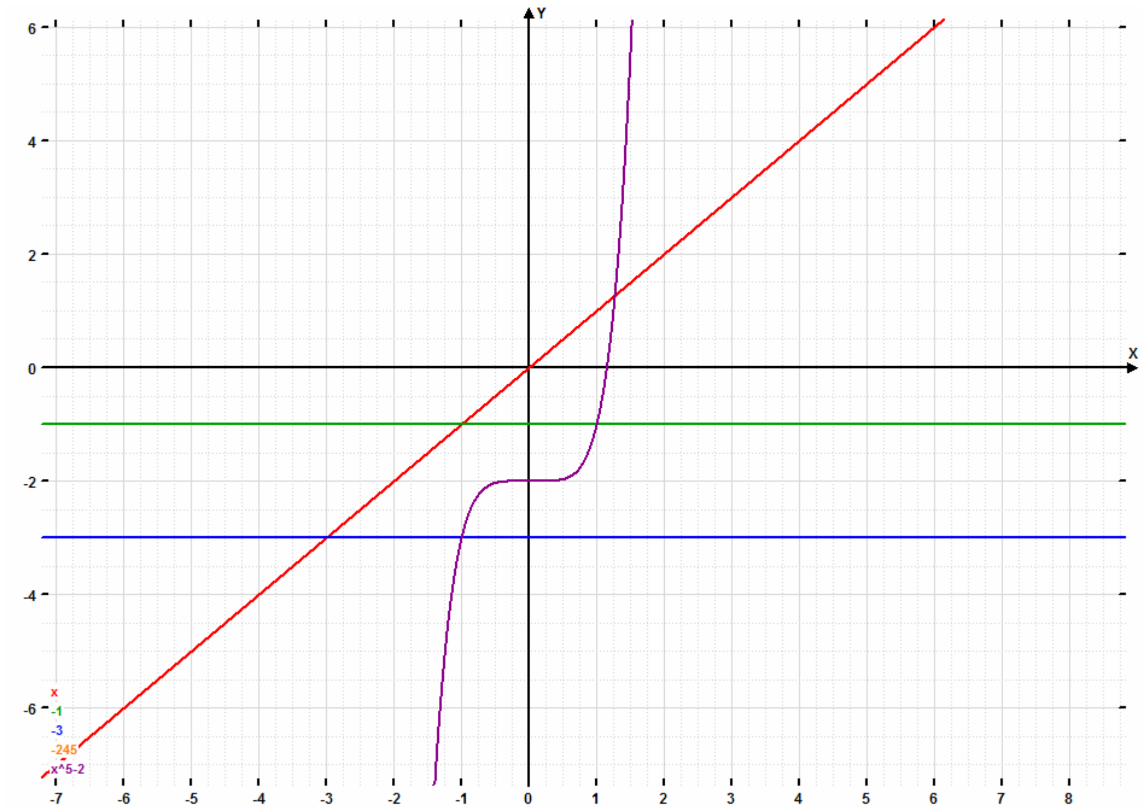
$n = 1 \rightarrow x_2 = 1/(x_1^2 - 1) \rightarrow x_2 = 1/(2^2 - 1) \rightarrow x_2 = 1/3 \approx 0.333$
$n = 2 \rightarrow x_3 = 1/(x_2^2 - 1) \rightarrow x_3 = 1/((1/3)^2 - 1) \rightarrow x_3 = -9/8 \approx -1.125$
$n = 3 \rightarrow x_4 = 1/(x_3^2 - 1) \rightarrow x_4 = 1/((-9/8)^2 - 1) \rightarrow x_4 = 64/17 \approx 3.365$
$n = 4 \rightarrow x_5 = 1/(x_4^2 - 1) \rightarrow x_5 = 1/((64/17)^2 - 1) \rightarrow x_5 = 289/3807 \approx 0.076$
$n = 5 \rightarrow x_6 = 1/(x_5^2 - 1) \rightarrow x_6 = 1/((289/3807)^2 - 1) \rightarrow x_6 \approx -1.006$

Unfortunately, this did not end up working at all, and the graph did not show any kind of pattern, relationships between the lines, or signs of convergence.

c) Repeat steps (a) and (b) for the following equations: (i) $x^5 - x - 2 = 0$ (ii) $x^3 - 2x - 4 = 0$

(i) Add x to both sides of the equation to get $x^5 - 2 = x$. This would result it to be $g(x_n) = x_{n+1} = x_n^5 - 2$, where $x_1 = 1$. The results would then be as follows.

$n = 1 \rightarrow x_2 = x_1^5 - 2 \rightarrow x_2 = 1^5 - 2 = -1$ $n = 2 \rightarrow x_3 = x_2^5 - 2 \rightarrow x_3 = -1^5 - 2 = -3$
$n = 3 \rightarrow x_4 = x_3^5 - 2 \rightarrow x_4 = -3^5 - 2 = -245$



And yet again, the ladder pattern appears, and although the graph does not show all five iterations, the two x values that have been graphed are sufficient to show that the ladder pattern will continue, and that the points diverge instead of converge.

Now, in order to rewrite the equation again, we add 2 to both sides of the equation to make it $x^5 - x = 2$, and then factor out an x on the left side of the equation, giving us $x(x^4-1) = 2$. Finally, we divide both sides by the quantity (x^4-1) to make $x = 2/(x^4-1)$. Then, again create the iteration $x_n = 2/(x_n^4-1)$ using $x_1 = 2$.

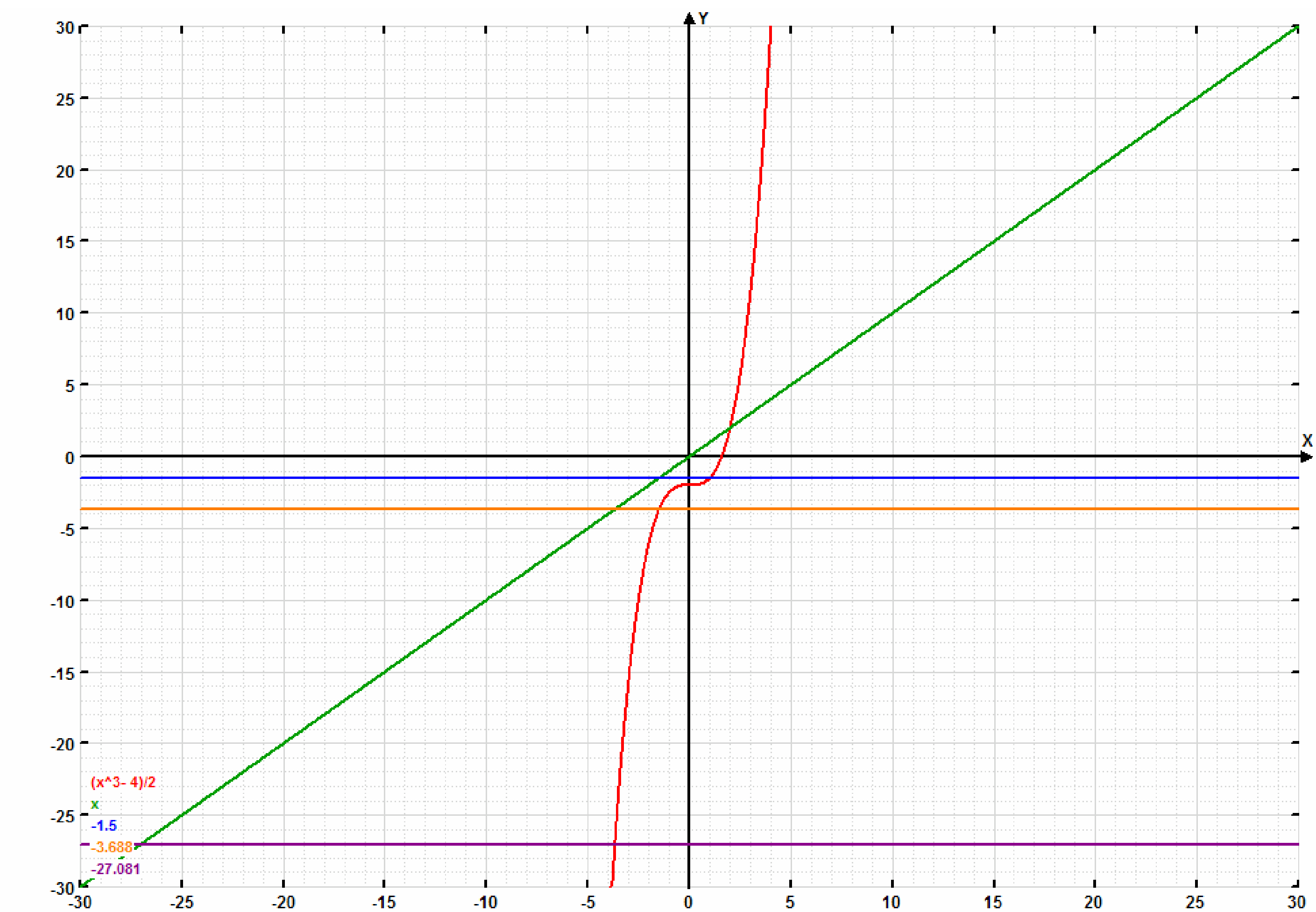
$n = 1 \rightarrow x_2 = 2/(x_1^4-1) \rightarrow x_2 = 2/(2^4-1) = 2/15 = 0.133$
$n = 2 \rightarrow x_3 = 2/(x_2^4-1) \rightarrow x_3 = 2/((2/15)^4-1) = -2.001$
$n = 3 \rightarrow x_4 = 2/(x_3^4-1) \rightarrow x_4 = 2/((-2.001)^4-1) = -0.117$
$n = 4 \rightarrow x_5 = 2/(x_4^4-1) \rightarrow x_5 = 2/((-0.117)^4-1) = -2.000$
$n = 5 \rightarrow x_6 = 2/(x_5^4-1) \rightarrow x_6 = 2/((-2)^4-1) = 0.133$

However, like it was the previous time, this iteration did not work and did not produce a graph that showed us the convergence.

ii) $x^3 - 2x - 4 = 0$. Add 2x to both sides of the equation to make it $x^3 - 4 = 2x$. Then, divide both sides by two, which will then make the equation $(x^3 - 4)/2 = x$. That is then changed into the iteration $x_{n+1} = (x_n^3 - 4)/2$, and we will solve this by using $x_1 = 1$.

$n = 1 \rightarrow x_2 = (x_1^3-4)/2 \rightarrow x_2 = (1^3 - 4)/2 = -3/2 = -1.5$
$n = 2 \rightarrow x_3 = (x_2^3-4)/2 \rightarrow x_3 = (-1.5^3 - 4)/2 = -3.688$
$n = 3 \rightarrow x_4 = (x_3^3-4)/2 \rightarrow x_4 = (-3.688^3 - 4)/2 = -27.081$

Like the previous two, this graph also contains a divergence, and the ladder-shape pattern.



So, to form a different iteration, we could add 2x + 4 on both sides to make the equation $x^3 = 2x + 4$. Then, the cubed root can be taken, to give the equation $x = (2x + 4)^{1/3}$. This would then lead to $x_{n+1} = (2x_n + 4)^{1/3}$, with using $x_1 = 1$.

$n = 1 \rightarrow x_2 = (2x_1 + 4)^{1/3} \rightarrow x_2 = (2 \cdot 1 + 4)^{1/3} = 1.817$

$n = 2 \rightarrow x_3 = (2x_2 + 4)^{1/3} \rightarrow x_3 = (2 \cdot 1.817 + 4)^{1/3} = 1.969$

$n = 3 \rightarrow x_4 = (2x_3 + 4)^{1/3} \rightarrow x_4 = (2 \cdot 1.969 + 4)^{1/3} = 1.995$

$n = 4 \rightarrow x_5 = (2x_4 + 4)^{1/3} \rightarrow x_5 = (2 \cdot 1.995+ 4)^{1/3} = 1.999$

$n = 5 \rightarrow x_6 = (2x_5 + 4)^{1/3} \rightarrow x_6 = (2 \cdot 1.999 + 4)^{1/3} = 2.000$